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ABSTRACT

A number of important control and estimation problems in the field of aerospace and avionics involve dynamical models that are quadratic and bilinear functions of system states and inputs. In this report, formal definitions of free and forced quadratic/bilinear dynamical systems are given and the algebraic structure of this class of systems is explored with a view to setting up a systematic approach for deriving state and measurement models. Properties of quadratic and bilinear vectors are investigated and relationships between these and linear vectors established. Systematic procedure for constructing quadratic state and bilinear state-input vectors is derived. The quadratic/bilinear vector modeling technique is applied to the formulation of a state-space model for a second order approximation of a general non-linear system.

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The Algebraic Structure of Quadratic and Bilinear Systems

Executive Summary

A number of important control and estimation problems in the field of aerospace and avionics involve dynamical models that are quadratic and bilinear functions of system states and inputs. Examples of some of these problems are: target tracking, vehicle navigation, guidance and autopilot design. While there are a number of different ways of posing (modeling) the above problems, it appears that by appropriate choice of state and input variables, systems involving translational and rotational kinematics can be modeled as a set of differential equations that contain linear plus quadratic and/or bilinear expressions. The well-known Riccati differential equation that occurs in the synthesis of optimal control and estimation problems is also of this class of differential equations.

In this report, formal definitions of free and forced quadratic/bilinear dynamical systems are given and the algebraic structure of this class of systems is explored with a view to setting up a systematic approach for deriving state and measurement models. This class of dynamical systems is characterized by a set of first order differential equations with the RHS that contains linear and quadratic terms in system states as well as bilinear terms involving state and input (or control) variables. Properties of quadratic and bilinear vectors are investigated and relationships between these and linear vectors are established. Systematic procedure for constructing quadratic state and bilinear state-input vectors is derived. The quadratic algebraic structure is applied in the formulation of a state-space model for a second order approximation of a general (analytic) non-linear system.

The concept of quadratic and bilinear generator matrices is introduced that allows linear (state and state-input) vectors to be mapped onto quadratic and bilinear vectors. Properties of these generator matrices are explored and it is shown that corresponding inverse generator matrices may be defined that allow quadratic and bilinear vectors to be mapped onto linear state and linear input vectors. State-space representation of the system output or measurement model is also derived for the case where this contains linear and quadratic terms in states and bilinear terms in state-disturbance (noise) variables.

The state-space structure of quadratic/bilinear dynamical systems considered in this report should facilitate analysis of this class of non-linear systems and possibly lead to general synthesis techniques or extension of linearised techniques to these problems.

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1. Introduction

A number of important control and estimation problems in the field of aerospace and avionics involve dynamical models that are quadratic and bilinear functions of system states and inputs. Examples of some of these problems are: target tracking, vehicle navigation and guidance and autopilot design [1-8]. While there are a number of different ways of posing (modeling) the above problems, it appears that by appropriate choice of state and input variables, systems involving tanslational and rotational kinematics can be modeled as a set of differential equations that contain linear plus quadratic and/or bilinear expressions. The well-known Riccati differential equation [9,10] that occurs in the synthesis of optimal control and estimation problems is also of this class of differential equations.

Other applications of bilinear systems are in the field of industrial process control, economics, biological systems, and chemical process modelling and control [11-13]. In addition to the applications, various researchers have also reported on important developments in bilinear systems theory and related stabilization, observability and controllability issues, including the Lie-algebra approach [14-16]. The control synthesis problem, involving bilinear systems, has been addresses by a number of authors utilising the optimum control theory [17-20]. While the above research lays down a fundamental and, in many respects, rigorous basis, for system analysis and synthesis for this class of non-linear systems, there still remains a need to apply the results of these investigations to high dimension systems (involving large number of state and control variables) that arise in many practical engineering problems. Motivated by this need, the current report addresses issues relating to the algebraic structure of quadratic and bilinear dynamical systems and proposes an approach for state-space formulation of the dynamical models that should allow for systematic and generalized procedures in dealing with practical control analysis and synthesis problems.

In this report, formal definitions of free and forced quadratic/bilinear dynamical systems are given and the algebraic structure of this class of systems is explored with a view to setting up a systematic approach for deriving state dynamics and measurement models. This class of dynamical systems is characterized by a set of first order differential equations with the RHS that contains linear and quadratic terms in system states as well as bilinear terms involving state and input (or control) variables. Properties of quadratic and bilinear vectors are investigated and relationships between these and linear vectors are established. Systematic procedure for constructing quadratic state and bilinear state-input vectors is derived. The quadratic algebraic structure, developed in this report, is applied in the formulation of a state-space model for a second order approximation to a general (analytic) non-linear system.

The concept of quadratic and bilinear generator matrices is introduced that allows linear (state and state-input) vectors to be mapped onto quadratic and bilinear vectors. Properties of these generator matrices are explored and it is shown that corresponding inverse generator matrices may be defined that allow quadratic and bilinear vectors to be mapped onto linear state and linear input vectors. State-space representation of the system output or measurement model is also derived for the case where this contains linear and quadratic terms in states and bilinear terms in state-disturbance (noise) variables.

The state-space structure of the quadratic/bilinear dynamical systems considered in this report should facilitate analysis of this class of non-linear systems and possibly lead to general synthesis techniques or extension of linearised techniques for applications to these problems.

The algebraic structure of free quadratic dynamical systems is considered in section 2 of this report and includes consideration of various properties of quadratic state vectors and their relationships with linear state vectors. Bilinear vectors are considered in section 3 along with the properties of these vectors and their relationships with linear vectors. Sections 4 and 5 consider forced dynamical systems and measurement models where the latter includes linear as well as quadratic terms in state variables and bilinear terms involving state and disturbance variables. In section 6 general non-linear dynamical systems with analytic non-linearity is considered and a second order perturbation state-space model of the system is derived in a quadratic/bilinear vector form.

2. Free Dynamical System

In this section we formalise the definition of a class of non-linear dynamical systems whose time evolution of states may be expressed via a set of first order differential equations, the right hand side (RHS) of which contain both linear and quadratic state terms. A formal definition will lead to a state vector characterisation (or a state-space representation) of these systems. The class of dynamical systems being considered in this section will be referred to as free (or uncontrolled) quadratic dynamical systems.

2.1. Definition

An n-state non-linear dynamical system will be defined as a *"free quadratic dynamical system"* (free-QDS) if its dynamics can be expressed by the following set of differential equations:

$$\frac{d}{dt}x_i(t) = \sum_{j=1}^{n} a_{i,j}(t)x_j(t) + \sum_{k=1}^{n}\sum_{j=k}^{n} q_{k,j}^{[i]}(t)x_k(t)x_j(t); \quad i = 1,2,\ldots,n \tag{2-1}$$

Where:

$\frac{d}{dt}(\cdot)$: is the time-derivative

$\{x_j(t);\quad j = 1,2,3,\ldots,n\}$: are linear system states

$\{x_k(t)x_j(t);\quad k = 1,2,3,\ldots,n;\quad j = k+1, k+2,\ldots,n\}$: are quadratic system states

$\{a_{i,j}(t);\quad i = 1,2,3,\ldots,n;\ j = 1,2,3,\ldots,n\}$: are linear system state coefficients

$\{q_{k,j}^{[i]}(t);\quad i = 1,2,3,\ldots,n;\ k = 1,2,3,\ldots,n;\ j = k,k+1,,\ldots,n\}$: are quadratic system state coefficients

Alternatively, the free-QDS of equation (2-1) may also be written as:

$$\frac{d}{dt}x_i(t) = \sum_{j=1}^{n} a_{i,j}(t)x_j(t) + \sum_{k=1}^{n}\sum_{j=k}^{n} \alpha_{k,j} b_{i,l}(t)x_k(t)x_j(t);\quad i = 1,2,\ldots,n \tag{2-2}$$

Where:

$$l = (k-1)n + j - \frac{k(k-1)}{2} = (k-1)\left(n - \frac{k}{2}\right) + j$$

$$\alpha_{k,j} = \begin{cases} 1 & \text{if } k = j \\ \sqrt{2} & \text{if } k \neq j \end{cases}$$

$$q_{k,j}^{[i]}(t) = \alpha_{k,j} b_{i,l}(t)$$

Remarks:

1. The RHS of equations (2-1) and (2-2) contain linear terms $\{x_i(t)\}$ as well as quadratic terms $\{x_k(t)x_j(t)\}$. The coefficients of the quadratic terms $\{q_{k,j}^{[i]}\}$ are related to the elements of a symmetric matrix $\left[Q^{[i]*}\right]$ that may be used to generate the scalar quadratic function $\left\{\underline{x}^T\left[Q^{[i]*}\right]\underline{x}\right\}$ from a *"linear state vector"* (*lsv*): $\underline{x} = [x_1\quad x_2\quad \ldots.\quad x_n]^T$ [1,2]. Here:

$$q_{k,j}^{[i]} = \beta_{k,j}\left(q_{k,j}^{[i]*} + q_{j,k}^{[i]*}\right)$$

$$\beta_{k,j} = \begin{cases} \frac{1}{2} & \text{if } k = j \\ 1 & \text{if } k \neq j \end{cases}$$

2. The RHS of equation (2-2) is identical to that of equation (2-1) except for the coefficients of the quadratic terms. The terms $\{q_{k,j}^{[i]}\}$ have been replaced by the terms $\{\alpha_{k,j} b_{i,l}(t)\}$ in equation (2-2). This substitution has been made to allow for state differential equations to be expressed in matrix notation, as will be seen later. The reason for introducing the normalisation factor $\{\alpha_{k,j} = \sqrt{2} \quad or \quad 1\}$ is explained later in this section.

3. In expanded form the differential equation (2-2) may be written as:

$$\frac{d}{dt}x_i(t) = a_{i,1}(t)x_1(t) + a_{i,2}(t)x_2(t) + \ldots. + a_{i,n}(t)x_n(t) + b_{i,1}(t)x_1^2(t) + \sqrt{2}b_{i,2}(t)x_1(t)x_2(t)$$
$$+\sqrt{2}b_{i,3}(t)x_1(t)x_3(t) + \ldots. + \sqrt{2}b_{i,n}(t)x_1(t)x_n(t) + b_{i,n+1}(t)x_2^2(t) + \sqrt{2}b_{i,n+2}(t)x_2(t)x_3(t)$$
$$+\sqrt{2}b_{i,n+3}(t)x_2(t)x_4(t) + \ldots. + \sqrt{2}b_{i,2n-1}(t)x_2(t)x_n(t) + \ldots. + b_{i,\left[\frac{n(n+1)}{2}-5\right]}(t)x_{n-2}^2(t)$$
$$+\sqrt{2}b_{i,\left[\frac{n(n+1)}{2}-4\right]}(t)x_{n-2}(t)x_{n-1}(t) + \sqrt{2}b_{i,\left[\frac{n(n+1)}{2}-3\right]}(t)x_{n-2}(t)x_n(t) + b_{i,\left[\frac{n(n+1)}{2}-2\right]}(t)x_{n-1}^2(t)$$
$$+\sqrt{2}b_{i,\left[\frac{n(n+1)}{2}-1\right]}(t)x_{n-1}(t)x_n(t) + b_{i,\left[\frac{n(n+1)}{2}\right]}(t)x_n^2(t)$$

for $\{i = 1,2,\ldots.,n\}$ (2-3)

4. Note that, in the quadratic part of the above equations, there are $\left\{\frac{n(n+1)}{2}\right\}$ coefficients $\{b_{i,l}(t)\}$, for each i, associated with the quadratic terms $\{x_k(t)x_j(t)\}$.

5. The introduction of the factor $\sqrt{2}$ in the coefficients was suggested in [3], and as will be shown later, it lends a useful property to the "metric" for a *"quadratic state vector" (qsv)*. However, before we discuss this, it will be instructive to consider an example of a second order free-QDS. The argument (t) is suppressed in the sequel, it being understood that variables and coefficients are functions of time unless otherwise stated.

2.2. Example

This example has been selected in order explore some useful properties of a *qsv* and relate these to the more familiar properties of an *lsv*. We consider a free-QDS for n=2; in this case the system differential equations may be written as:

$$\frac{d}{dt}x_1 = a_{1,1}x_1 + a_{1,2}x_2 + b_{1,1}x_1^2 + \sqrt{2}b_{1,2}x_1x_2 + b_{1,3}x_2^2$$
$$\frac{d}{dt}x_2 = a_{2,1}x_1 + a_{2,2}x_2 + b_{2,1}x_1^2 + \sqrt{2}b_{2,2}x_1x_2 + b_{2,3}x_2^2 \qquad (2\text{-}4)$$

Or, in matrix notation, differential equation (2-4) may written as:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \end{bmatrix}\begin{bmatrix} x_1^2 \\ \sqrt{2}x_1x_2 \\ x_2^2 \end{bmatrix} \qquad (2\text{-}5)$$

Equation (2-5) is of the form:

$$\frac{d}{dt}\underline{x}^{[1]} = \left[A^{[1]}\right]\underline{x}^{[1]} + \left[B^{[2]}\right]\underline{x}^{[2]} \qquad (2\text{-}6)$$

Where:

$\underline{x}^{[1]} = [x_1 \quad x_2]^T$: is a $[2\times 1]$ linear state vector (*lsv*)

$\underline{x}^{[2]} = \left[x_1^2 \quad \sqrt{2}x_1x_2 \quad x_2^2\right]^T$: is a $[3\times 1]$ quadratic state vector (*qsv*)

$\left[A^{[1]}\right] = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$: is a $[2\times 2]$ linear state coefficient matrix

$\left[B^{[2]}\right] = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \end{bmatrix}$: is a $[2\times 3]$ quadratic state coefficient matrix

Remarks:

1. For this example, the quadratic state vector *(qsv)* $\left\{\underline{x}^{[2]}\right\}$ may be regarded as a vector spanning the space R^3 whose elements are the terms of a homogeneous quadratic polynomial in $\{x_1, \quad x_2\}$ arranged in lexicographic order.

2. The inclusion of the factor $\sqrt{2}$ imparts a desirable property to the metric of the *qsv*. Thus, analogous to the definition of a metric for an *lsv* let us define a metric for a *qsv* as the square root of the sum of squares of the quadratic elements. That is, for n=2:

$$\left\| \underline{x}^{[2]} \right\| = \left\{ \sum_{1}^{3} \left(x_i^{[2]}\right)^2 \right\}^{\frac{1}{2}} = \left\{ \left(x_1^2\right)^2 + \left(\sqrt{2}x_1x_2\right)^2 + \left(x_2^2\right)^2 \right\}^{\frac{1}{2}} = \left\{x_1^2 + x_2^2\right\} = \left\| \underline{x}^{[1]} \right\|^2 \qquad (2\text{-}7)$$

3. It is interesting to note that by defining the *qsv* as we have done above, and by adhering to the familiar definition of a metric in a linear vector space, a relationship

between the metrics of the *qsv* and the *lsv* may be established. That is, the norm of the *qsv* is equal to the square of the norm of the associated *lsv*.

4. A *qsv* formed by including the factor $\sqrt{2}$ will be referred to as a 'normalised' *qsv* or simply as a *qsv*. The 'non-normalised' *qsv*, that is $\underline{x}^{[2]} = \lfloor x_1^2 \quad x_1x_2 \quad x_2^2 \rfloor$ can also be used in the formulation of a free-QDS. However, in this case:

$$\left\|\underline{x}^{[2]}\right\| = \left\{ \left(x_1^2\right)^2 + (x_1x_2)^2 + \left(x_2^2\right)^2 \right\}^{\frac{1}{2}} \tag{2-8}$$

5. We observe that a second order quadratic polynomial of the form:

$$p\left(\underline{x}^{[1]}\right) = x_1^2 + \sqrt{2}x_1x_2 + x_2^2 \tag{2-9}$$

whose terms form elements of the *qsv* $\underline{x}^{[2]}$, may be expressed as a combinations of scalar quadratic functions (forms) using the *lsv* $\underline{x}^{[1]}$ as follows:

$$\begin{aligned} p\left(\underline{x}^{[1]},\underline{x}^{[1]}\right) &= \left(\underline{x}^{[1]}\right)^T \left[E_{1,1}\right]\left(\underline{x}^{[1]}\right) + \left(\underline{x}^{[1]}\right)^T \left[E_{1,2}\right]\left(\underline{x}^{[1]}\right) + \left(\underline{x}^{[1]}\right)^T \left[E_{2,2}\right]\left(\underline{x}^{[1]}\right) \\ &= \left(\underline{x}^{[1]}\right)^T \left\{\sum_{k=1}^{2}\sum_{j=k}^{2}\left[E_{k,j}\right]\right\}\left(\underline{x}^{[1]}\right) \end{aligned} \tag{2-10}$$

Where:

$$\left[E_{1,1}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad \left[E_{1,2}\right] = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \left[E_{2,2}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{2-11}$$

6. We have selected these $\lfloor E_{k,j} \rfloor$ matrices to be symmetric. These matrices serve as the basis for the quadratic polynomial; in fact, any second order quadratic polynomial may be written (in terms of a linear combination of matrices such as $\lfloor E_{k,j} \rfloor$) as follows:

$$p'\left(\underline{x}^{[1]},\underline{x}^{[1]}\right) = \left(\underline{x}^{[1]}\right)^T \left\{\sum_{k=1}^{2}\sum_{j=k}^{2}\lambda_l\left[E_{k,j}\right]\right\}\left(\underline{x}^{[1]}\right); \quad \lambda_l = a\ scalar \tag{2-12}$$

$$l = (k-1)\left(n - \frac{k}{2}\right) + j$$

7. The basis matrices $\lfloor E_{k,j} \rfloor$ may also be used to construct the *qsv* from the *lsv*, which allows us to establish a relationship between the *qsv* and the associated *lsv* as follows:

$$\underline{x}^{[2]} = \begin{bmatrix} x_1^{\;2} \\ \sqrt{2}x_1x_2 \\ x_2^{\;2} \end{bmatrix} = \begin{bmatrix} \underline{x}^T[E_{1,1}]\underline{x} \\ \underline{x}^T[E_{1,2}]\underline{x} \\ \underline{x}^T[E_{2,2}]\underline{x} \end{bmatrix} = \begin{bmatrix} \underline{x}^T[E_{1,1}] \\ \underline{x}^T[E_{1,2}] \\ \underline{x}^T[E_{2,2}] \end{bmatrix}[\underline{x}] = \begin{bmatrix} x_1 & 0 \\ \dfrac{x_2}{\sqrt{2}} & \dfrac{x_1}{\sqrt{2}} \\ 0 & x_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left[X_{[1]}^{[2]}\right]\underline{x}^{[1]} \tag{2-13}$$

Where:

$$\left[X_{[1]}^{[2]}\right] = \begin{bmatrix} x_1 & 0 \\ \dfrac{x_2}{\sqrt{2}} & \dfrac{x_1}{\sqrt{2}} \\ 0 & x_2 \end{bmatrix} = \begin{bmatrix} \underline{x}^T[E_{1,1}] \\ \underline{x}^T[E_{1,2}] \\ \underline{x}^T[E_{2,2}] \end{bmatrix} = \left[[E_{1,1}]\underline{x} \quad [E_{1,2}]\underline{x} \quad [E_{2,2}]\underline{x}\right]^T \tag{2-14}$$

: is a $[3 \times 2]$ matrix

8. The matrix $\left[X_{[1]}^{[2]}\right]$, which will be referred to as the *"quadratic generator matrix" (qgm)*, maps the *lsv* $\underline{x}^{[1]}$ onto the associated *qsv* $\underline{x}^{[2]}$. It will be shown later that provided $\underline{x}^{[1]} \neq \underline{0}$, there exists the *"inverse quadratic generator matrix" (iqgm)*: $\left[X_{[2]}^{[1]}\right] = \left[X_{[1]}^{[2]}\right]^{-1}$ that maps the *qsv* $\underline{x}^{[2]}$ onto the associated *lsv* $\underline{x}^{[1]}$. (Note the change between the sub and super-scripts between *qgm* and *iqgm* symbols). This inverse relationship follows from equation (2-13):

Now, $\underline{x}^{[2]} = \left[X_{[1]}^{[2]}\right]\underline{x}^{[1]}$

$$\Rightarrow \quad \left[X_{[1]}^{[2]}\right]^T \underline{x}^{[2]} = \left[X_{[1]}^{[2]}\right]^T \left[X_{[1]}^{[2]}\right]\underline{x}^{[1]}$$

$$\Rightarrow \quad \left\{\left[X_{[1]}^{[2]}\right]^T \left[X_{[1]}^{[2]}\right]\right\}^{-1} \left[X_{[1]}^{[2]}\right]^T \underline{x}^{[2]} = \underline{x}^{[1]}$$

$$i.e: \underline{x}^{[1]} = \left[X_{[2]}^{[1]}\right]\underline{x}^{[2]} \tag{2-15}$$

Where:

$\left[X_{[2]}^{[1]}\right] = \left\{\left[X_{[1]}^{[2]}\right]^T \left[X_{[1]}^{[2]}\right]\right\}^{-1} \left[X_{[1]}^{[2]}\right]^T$: will be referred to as an inverse quadratic generator matrix *iqgm*.

In fact, using equation (2-14) and after some algebraic manipulation we get:

$$\left\{\left[X_{[1]}^{[2]}\right]^T \left[X_{[1]}^{[2]}\right]\right\} = \begin{bmatrix} \left(x_1^{\;2} + \dfrac{x_2^{\;2}}{2}\right) & \left(\dfrac{x_1x_2}{2}\right) \\ \left(\dfrac{x_1x_2}{2}\right) & \left(\dfrac{x_1^{\;2}}{2} + x_2^{\;2}\right) \end{bmatrix} \tag{2-16}$$

and the $[2\times 3]$ *iqgm* is given by:

$$\left[X_{[2]}^{[1]}\right]=\frac{1}{\Delta}\begin{bmatrix}\left\{x_1\left(\frac{x_1^2}{2}+x_2^2\right)\right\} & \left\{\frac{x_2}{\sqrt{2}}\left(\frac{x_1^2}{2}+x_2^2\right)-\frac{x_1^2x_2}{2\sqrt{2}}\right\} & \left\{-\frac{x_1x_2^2}{2}\right\}\\ \left\{-\frac{x_1^2x_2}{2}\right\} & \left\{\frac{x_1}{\sqrt{2}}\left(x_1^2+\frac{x_2^2}{2}\right)-\frac{x_1x_2^2}{2\sqrt{2}}\right\} & \left\{x_2\left(x_1^2+\frac{x_2^2}{2}\right)\right\}\end{bmatrix} \tag{2-17}$$

With: $\Delta=\frac{\left(x_1^2+x_2^2\right)^2}{2}$

Note: that $\Delta\neq 0;\quad provided\ \underline{x}\neq\underline{0}$.

9. The results of this example are generalized in the sequel; however before we leave this example, we shall use equation (2-13) or the *qgm*, to give an alternative state space representation of the free-QDS analogous of equation (2-6). That is:

$$\frac{d}{dt}\underline{x}^{[1]}=\left\{\left[A^{[1]}\right]+\left[B^{[2]}\right]\left[X_{[1]}^{[2]}\right]\right\}\underline{x}^{[1]} \tag{2-18}$$

2.3. State-Space Representation of a General n-State Free-QDS:

In vector notation a general n-state free-QDS [see equation (2-6), Example 2.2] may be written as:

$$\frac{d}{dt}\underline{x}^{[1]}=\left[A^{[1]}\right]\underline{x}^{[1]}+\left[B^{[2]}\right]\underline{x}^{[2]} \tag{2-19}$$

Where:

$\underline{x}^{[1]}=\left[x_1\quad x_2\quad x_3\quad \ldots\quad x_n\right]^T$: is an $[n\times 1]$ linear state vector (*lsv*)

$\underline{x}^{[2]}=\left[x_1^2\quad \sqrt{2}x_1x_2\quad \sqrt{2}x_1x_3\quad \ldots\quad \sqrt{2}x_1x_n\quad x_2^2\quad \sqrt{2}x_2x_3\quad \sqrt{2}x_2x_4\quad \ldots\quad \sqrt{2}x_2x_n\quad \ldots\right.$
$\left.\quad x_3^2\quad \ldots\quad \sqrt{2}x_{n-2}x_n\quad x_{n-1}^2\quad \sqrt{2}x_{n-1}x_n\quad x_n^2\right]^T$

: is an $\left[\frac{n(n+1)}{2}\times 1\right]$ quadratic state vector (*qsv*)

Note that the l^{th} term $\left\{x_l^{[2]}\right\}$ of the vector $\underline{x}^{[2]}$ is given by:

$$x_l^{[2]}=\alpha_{k,j}x_kx_j;\quad k=1,\ldots,n;\ j=k,k+1,k+2,\ldots,n$$

$$l=(k-1)\left(n-\frac{k}{2}\right)+j;\quad \alpha_{k,j}=\begin{cases}1 & if\ k=j\\ \sqrt{2} & if\ k\neq j\end{cases}$$

$$\left[A^{[1]}\right]=\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{bmatrix} \text{: is an } [n\times n] \text{ linear state coefficient matrix}$$

$$\left[B^{[2]}\right]=\begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \cdots & b_{1,\left[\frac{(n+2)(n-1)}{2}\right]} & b_{1,\left[\frac{n(n+1)}{2}\right]} \\ b_{2,1} & b_{2,2} & b_{2,3} & \cdots & b_{2,\left[\frac{(n+2)(n-1)}{2}\right]} & b_{2,\left[\frac{n(n+1)}{2}\right]} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n,1} & b_{n,2} & b_{n,3} & \cdots & b_{n,\left[\frac{(n+2)(n-1)}{2}\right]} & b_{n,\left[\frac{n(n+1)}{2}\right]} \end{bmatrix} \text{: is an } \left[n \, x \, \frac{n(n+1)}{2}\right] \text{ quadratic-state}$$

coefficient matrix

Utilising the concept of the *qgm* introduced in Example (2-2), we may write an alternative form of a general n-state free-QDS as:

$$\frac{d}{dt}\underline{x}^{[1]} = \left\{\left[A^{[1]}\right]+\left[B^{[2]}\right]\left[X^{[2]}_{[1]}\right]\right\}\underline{x}^{[1]} \tag{2-20}$$

Where:

$\left[X^{[2]}_{[1]}\right]$: is the $\left[\frac{n(n+1)}{2}\times n\right]$ *qgm* defined in equation (B1-1), Appendix-B.

Remarks:

1. We make the observation that provided $\underline{x}\neq\underline{0}$, none of the columns of a *qgm* $\left[X^{[2]}_{[1]}\right]$ equation (B2-1) may be expressed as linear combinations of the other (n-1) columns; that is, it has a rank=n, and hence the matrix $\left\{\left[X^{[2]}_{[1]}\right]^T\left[X^{[2]}_{[1]}\right]\right\}$ is of full rank and therefore invertible.

2. The *qgm* for a general n-state system may be constructed using the basis matrices introduced earlier [see equations (2-10) to (2-14)]. It follows from equation (2-13) that:

$$\begin{aligned}\left[X^{[2]}_{[1]}\right]=\Big[&[E_{1,1}]\underline{x} \quad [E_{1,2}]\underline{x} \quad [E_{1,3}]\underline{x} \quad \cdots \quad [E_{1,n}]\underline{x} \quad [E_{2,2}]\underline{x} \quad [E_{2,3}]\underline{x} \quad [E_{2,4}]\underline{x} \quad \cdots \\ &[E_{2,n}]\underline{x} \quad [E_{3,3}]\underline{x} \quad \cdots \quad [E_{n-2,n}]\underline{x} \quad [E_{n-1,n-1}]\underline{x} \quad [E_{n-1,n}]\underline{x} \quad [E_{n,n}]\underline{x}\Big]^T\end{aligned} \tag{2-21}$$

Where:

$$[E_{1,1}]=\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix};[E_{1,2}]=\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix};\dots;[E_{1,n}]=\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix};$$

$$[E_{2,2}]=\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix};[E_{2,3}]=\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix};\dots;[E_{n,n}]=\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Note that the basis matrices $[E_{k,j}]$ have elements given by:

$$\{E_{k,j}\}_{r,s}=\begin{cases} \text{for} \quad j=k & \begin{cases} 1 & \text{for} \quad r=s \\ 0 & \text{otherwise} \end{cases} \\ \text{for} \quad j\neq k & \begin{cases} \frac{1}{\sqrt{2}} & \text{for} \quad r=k,s=j \text{ and } r=j,s=k \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

2.4. Properties of the General n-State Quadratic Generator Matrix (qgm):

It can easily be verified that for the n-state *qsv* and the corresponding *qgm*, the following properties hold (the proof can be arrived at in a manner similar to that of the 2-state case, and is left to the reader):

2.4.1. Metric Property:

$$\left\| \underline{x}^{[2]} \right\| = \left\{ \left(\sum_{l=1}^{\frac{n(n+1)}{2}} x_l^{[2]} \right)^2 \right\}^{\frac{1}{2}} = \left\{ \left(\sum_{\substack{k=1 \\ j=k}}^{n} \alpha_{k,j} x_k x_j \right)^2 \right\}^{\frac{1}{2}} = \left\| \underline{x}^{[1]} \right\|^2 \tag{2-22}$$

2.4.2. Mapping of lsv $\underline{x}^{[1]}$ onto qsv $\underline{x}^{[2]}$ through qgm $\left[X_{[1]}^{[2]}\right]$:

$$\underline{x}^{[2]} = \left[X_{[1]}^{[2]}\right]\underline{x}^{[1]} \tag{2-23}$$

2.4.3. *Inverse mapping of qsv $\underline{x}^{[2]}$ onto lsv $\underline{x}^{[1]}$ the iqgm $\left[X_{[2]}^{[1]}\right]$:*

$$\underline{x}^{[1]} = \left[X_{[2]}^{[1]}\right]\underline{x}^{[2]} \tag{2-24}$$

Where: $\left[X_{[2]}^{[1]}\right] = \left\{\left[X_{[1]}^{[2]}\right]^T \left[X_{[1]}^{[2]}\right]\right\}^{-1} \left[X_{[1]}^{[2]}\right]^T$

2.4.4. *The Bilinear Property:*

Given any two "linear vectors" $\underline{x}^{[1]}, \underline{u}^{[1]} \in R^n$, the quadratic generator matrix *(qgm)* $\left[X_{[1]}^{[2]}\right] \in R^{\frac{n(n+1)}{2} x\, n}$ defined as in equation (B2-1) will be called the *"bilinear generator matrix" (bgm)* when it operates on $\underline{u}^{[1]} \in R^n$ to yield the *"bilinear vector"(bv)* $\underline{\pi}^{[2]}\left(\underline{x}^{[1]}, \underline{u}^{[1]}\right)$. Similarly, the quadratic generator matrix *(qgm)* $\left[U_{[1]}^{[2]}\right] \in R^{\frac{n(n+1)}{2} x\, n}$ defined as in equation (B2-1), *(with x_i replaced by u_i)* will also be called the *"bilinear generator matrix" (bgm)* when it operates on $\underline{x}^{[1]} \in R^n$ to yield the bilinear vector $\underline{\pi}^{[2]}\left(\underline{u}^{[1]}, \underline{x}^{[1]}\right)$:

Where:

$$\underline{\pi}^{[2]}\left(\underline{x}^{[1]}, \underline{u}^{[1]}\right) = \underline{\pi}^{[2]}\left(\underline{u}^{[1]}, \underline{x}^{[1]}\right) =$$
$$\Big[x_1u_1 \quad \frac{1}{\sqrt{2}}(x_1u_2 + x_2u_1) \quad \frac{1}{\sqrt{2}}(x_1u_3 + x_3u_1) \quad .... \quad \frac{1}{\sqrt{2}}(x_1u_n + x_nu_1) \quad x_2u_2 \quad \frac{1}{\sqrt{2}}(x_2u_3 + x_3u_2)$$
$$.... \quad \frac{1}{\sqrt{2}}(x_2u_n + x_nu_2) \quad .... \quad \frac{1}{\sqrt{2}}(x_{n-2}u_n + x_nu_{n-2}) \quad x_{n-1}u_{n-1} \quad \frac{1}{\sqrt{2}}(x_{n-1}u_n + x_nu_{n-1}) \quad x_nu_n \Big] \tag{2-25}$$

The vector $\underline{\pi}^{[2]}(.,.)$ as defined above in (2-25), includes terms in sum of products of the elements $\{x_k, u_j; k = 1,2,....,n \quad and \quad j = k, k+1,....,n\}$ taken in lexicographic order. This vector will be referred to as the bilinear vector *(bv)* of the corresponding linear vectors $\{\underline{x}^{[1]}, \underline{u}^{[1]}\}$. Note: that the order of the arguments in $\underline{\pi}^{[2]}(.,.)$ is unimportant; i.e. $\{\underline{x}^{[1]}, \underline{u}^{[1]}\}$ are interchangeable. Also, since both $\{\underline{x}^{[1]}, \underline{u}^{[1]}\}$ are n-order vectors, we shall classify this *bv* as having the dimension $\left\{\frac{n(n+1)}{2} \times 1\right\}$. Clearly:

$$\underline{\pi}^{[2]}\left(\underline{x}^{[1]}, \underline{u}^{[1]}\right) = \left[X_{[1]}^{[2]}\right]\underline{u}^{[1]} = \left[U_{[1]}^{[2]}\right]\underline{x}^{[1]} = \underline{\pi}^{[2]}\left(\underline{u}^{[1]}, \underline{x}^{[1]}\right) \tag{2-26}$$

Remarks:

1. In view of the properties (2.4.2) and (2.4.3) above, we shall formally recognize the *bgms* $\left[X_{[1]}^{[2]}\right],\left[U_{[1]}^{[2]}\right]\in R^{\frac{n(n+1)}{2}x\,n}$ as transformation matrices that define the mappings:

   a. $\left[X_{[1]}^{[2]}\right]:\underline{u}^{[1]}\Rightarrow\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)\equiv\underline{\pi}^{[2]}\left(\underline{u}^{[1]},\underline{x}^{[1]}\right)$

   b. $\left[U_{[1]}^{[2]}\right]:\underline{x}^{[1]}\Rightarrow\underline{\pi}^{[2]}\left(\underline{u}^{[1]},\underline{x}^{[1]}\right)\equiv\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)$

   c. $\left[X_{[1]}^{[2]}\right]:\underline{x}^{[1]}\Rightarrow\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{x}^{[1]}\right)\equiv\underline{x}^{[2]}$

   d. It can easily be verified that the norm of the bilinear vector satisfies the relationship:

$$\left\|\underline{\pi}^{[2]}\left(\underline{u}^{[1]},\underline{x}^{[1]}\right)\right\|\equiv\left\|\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)\right\|=\frac{1}{2}\left[\left(\underline{x}^{[1]T}\underline{u}^{[1]}\right)^{2}+\left\|\underline{x}^{[1]}\right\|^{2}\left\|\underline{u}^{[1]}\right\|^{2}\right]$$
$$=\frac{1}{2}\left\|\underline{x}^{[1]}\right\|^{2}\left\|\underline{u}^{[1]}\right\|^{2}\left(1+\cos^{2}\gamma\right)$$

   Where: γ is the angular separation between the vectors $\underline{x}^{[1]}$ *and* $\underline{u}^{[1]}$.

2. It will be instructive at this stage to consider a bilinear vector where the dimensions of the two linear vectors involved are different. Accordingly, given any two linear vectors $\underline{x}^{[1]}\in R^{n},\underline{u}^{[1]}\in R^{m};m\leq n$, the corresponding bilinear generator matrices *(bgms)* are given by:

$$\left[\overline{X}_{[1]}^{[2]}\right]_{l x m}\in R^{l x\,m},\left[\overline{U}_{[1]}^{[2]}\right]_{l x n}\in R^{l x\,n};l=\frac{m(2n-m+1)}{2}$$

   These matrices are defined in equations (B1-2) and (B1-3), Appendix B.

   This result along with the others given in this section will be used for developing the concept of a *"bilinear state-input vector" (bsiv)* and of forced (or controlled)-quadratic/bilinear dynamical systems (Forced-QBDS).

2.4.4.1. Example

For n=3, m=2, the *bgms* are given by the following relationships:

$$\left[\overline{X}_{[1]}^{[2]}\right]_{5x2}=\begin{bmatrix} x_1 & 0 \\ \frac{x_2}{\sqrt{2}} & \frac{x_1}{\sqrt{2}} \\ \frac{x_3}{\sqrt{2}} & 0 \\ 0 & x_2 \\ 0 & \frac{x_3}{\sqrt{2}} \end{bmatrix};\quad \left[\overline{U}_{[1]}^{[2]}\right]_{5x3}=\begin{bmatrix} u_1 & 0 & 0 \\ \frac{u_2}{\sqrt{2}} & \frac{u_1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{u_1}{\sqrt{2}} \\ 0 & u_2 & 0 \\ 0 & 0 & \frac{u_2}{\sqrt{2}} \end{bmatrix}$$

and

$$\begin{aligned} &\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\left[\overline{X}_{[1]}^{[2]}\right]_{2x5}\underline{u}^{[1]}=\left[\overline{U}_{[1]}^{[2]}\right]_{3x5}\underline{x}^{[1]}=\underline{\pi}^{[2]}\left(\underline{u}^{[1]},\underline{x}^{[1]}\right) \\ &=\left(x_1u_1 \quad \frac{1}{\sqrt{2}}(x_2u_1+x_1u_2) \quad \frac{1}{\sqrt{2}}(x_3u_1) \quad x_2u_2 \quad \frac{1}{\sqrt{2}}(x_3u_2)\right) \end{aligned} \tag{2-27}$$

Remarks

1. The *"bilinear n-state, m-input vector" (bsiv)* is given by:

$$\begin{aligned} &\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\underline{\pi}^{[2]}\left(\underline{u}^{[1]},\underline{x}^{[1]}\right)= \\ &\left[x_1u_1 \quad \frac{1}{\sqrt{2}}(x_1u_2+x_2u_1) \quad \frac{1}{\sqrt{2}}(x_1u_3+x_3u_1) \quad \ldots \quad \frac{1}{\sqrt{2}}(x_nu_1) \quad x_2u_2\right. \\ &\left.\frac{1}{\sqrt{2}}(x_2u_3+x_3u_2) \quad \ldots \quad \frac{1}{\sqrt{2}}(x_nu_2) \quad \ldots \quad \frac{1}{\sqrt{2}}(x_nu_{m-1}) \quad x_mu_m \quad \ldots \quad \frac{1}{\sqrt{2}}(x_nu_m)\right] \end{aligned} \tag{2-28}$$

2. Finally, we end this section by giving a block diagram for a free-QDS as reflected by the state vector equation (2-19). This is given in Figures 2.1.

3. For sake of convenience the sub-scripts outside the square brackets of the *bgms* have been dropped, that is:

$$\left[\overline{X}_{[1]}^{[2]}\right]_{lxm}\equiv\left[\overline{X}_{[1]}^{[2]}\right] \text{ and } \left[\overline{U}_{[1]}^{[2]}\right]_{lxm}\equiv\left[\overline{U}_{[1]}^{[2]}\right]$$

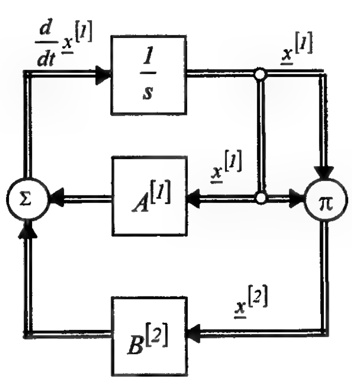

Figure 2.1 Block Diagram of the free-QDS of equation (2-19)

3. Bilinear State-Input Vector

Before we consider forced-QBDS, we shall formalize the structure and definition of a *"bilinear state-input vector"* (*bsiv*) generated from two linear vectors, namely, the n-dimensional state vector $\underline{x}^{[1]} \in R^n$ and the m-dimensional input vector $\underline{u}^{[1]} \in R^m$. In order to develop this concept we shall utilize the familiar scalar bilinear forms and its various properties. Accordingly, we consider the following scalar bilinear form:

$$\Pi_i\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\sum_{k=1}^{n}\sum_{j=1}^{m}s_{k,j}^{[i]}x_k u_j=\sum_{k=1}^{m}\sum_{j=1}^{n}s_{j,k}^{[i]}x_j u_k \tag{3-1}$$

In a more familiar matrix notation equation (3-1) may be written as:

$$\Pi_i\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\left(\underline{x}^{[1]}\right)^T\left[S^{[i]}\right]\left(\underline{u}^{[1]}\right)=\left(\underline{u}^{[1]}\right)^T\left[S^{[i]}\right]^T\left(\underline{x}^{[1]}\right)=\sum_{k=1}^{m}\sum_{j=1}^{n}s_{j,k}^{[i]}x_j u_k \tag{3-2}$$

Where:

$\left[S^{\{i\}}\right]=\left\{s_{k,j}^{[i]} : k=1,2,...,m; j=1,2,...,n; \text{with } m\le n\right\}$: is an $[n\times m]$ matrix

$\underline{x}^{[1]}=\left\{x_j^{[1]} : j=1,2,...,n\right\}$: is an $[n\times 1]$ *lsv*

$\underline{u}^{[1]}=\left\{u_j^{[1]} : j=1,2,...,m\right\}$: is an $[m\times 1]$ *"linear input (or control) vector"* (*liv*)

By expanding the double summation in equation (3-2) and rearranging the terms, it is easily verified that:

$$\sum_{k=1}^{m}\sum_{j=1}^{n}s_{j,k}^{[i]}x_j u_k=\sum_{k=1}^{m}\sum_{j=k}^{n}\alpha_{k,j}\left\{s_{k,j}^{[i]}x_k u_j+s_{j,k}^{[i]}x_j u_k\right\} \tag{3-3}$$

Where:

$$\alpha_{k,j}=\begin{cases}\dfrac{1}{2} & \text{for} \quad j=k\\[2ex] 1 & \text{otherwise}\end{cases}$$

Since $u_j=0 \quad \text{for} \quad j>m$, we may w.l.o.g assume $s_{k,j}^{[i]}=0 \quad \text{for} \quad j>m$.

Equation (3-3) may be written as:

$$\sum_{k=1}^{m}\sum_{j=k}^{n}\alpha_{k,j}\left\{s_{k,j}^{[i]}x_k u_j+s_{j,k}^{[i]}x_j u_k\right\}=\sum_{k=1}^{m}\sum_{j=k}^{n}\alpha_{k,j}\left\{p_{k,j}^{[i]}\left(x_k u_j+x_j u_k\right)+q_{k,j}^{[i]}\left(x_k u_j-x_j u_k\right)\right\}$$
$$=\sum_{k=1}^{m}\sum_{j=k}^{n}\alpha_{k,j}p_{k,j}^{[i]}\left(x_k u_j+x_j u_k\right)+\sum_{k=1}^{m}\sum_{j=k}^{n}\alpha_{k,j}q_{k,j}^{[i]}\left(x_k u_j-x_j u_k\right) \tag{3-4}$$

Where:

$$p_{k,j}^{[i]} = \frac{1}{2}\left(s_{k,j}^{[i]} + s_{j,k}^{[i]}\right); \quad q_{k,j}^{[i]} = \frac{1}{2}\left(s_{k,j}^{[i]} - s_{j,k}^{[i]}\right)$$

Remarks:

1. The first summation term of equation (3-4) defines a symmetric bilinear form or *"sum bilinear form" (sbf)*, while the second term defines a skew-symmetric or the *"difference bilinear form" (dbf)* (also known as the oscillating bilinear form [4]).

2. If the bilinear form, equation (3-4), is solely symmetrical then:

   $$s_{j,k}^{[i]} = s_{k,j}^{[i]}; \quad p_{j,k}^{[i]} = s_{j,k}^{[i]} = s_{k,j}^{[i]}; \quad q_{j,k}^{[i]} = 0$$

   And:

   $$\Pi_i(.,.) = \sum_{k=1}^{m}\sum_{j=k}^{n} \alpha_{k,j} p_{k,j}^{[i]} \left(x_k u_j + x_j u_k\right)$$

For notational consistency and to facilitate development of the state-space model utilizing the normalized form (to be considered later), we shall write equation (3-4) as:

$$\Pi_i\left(\underline{x}^{[l]}, \underline{u}^{[l]}\right) = \sum_{k=1}^{m}\sum_{j=k}^{n} \beta_{k,j} d_{k,l}\left(x_k u_j + x_j u_k\right) + \sum_{k=1}^{m}\sum_{j=k}^{n} \beta_{k,j} d^*{}_{k,l}\left(x_k u_j - x_j u_k\right) \tag{3-5}$$

Where:

$$\beta_{k,j} = \begin{cases} 1 & for \quad j = k \\ \frac{1}{\sqrt{2}} & otherwise \end{cases}$$

$$d_{i,l} = \frac{\alpha_{k,j} p_{k,j}^{[i]}}{\beta_{k,j}}; \quad d^*{}_{i,l} = \frac{\alpha_{k,j} q_{k,j}^{[i]}}{\beta_{k,j}}$$

$$l = (k-1)\left(n - \frac{k}{2}\right) + j; \quad k = 1,2,..,m; \quad j = k+1, k+2, ..., n$$

Remarks:

1. The bilinear form of equation (3-5) may be regarded as consisting of a weighted sum of normalized symmetric elements and a weighted sum of normalized skew-symmetric elements. Terms $\{\beta_{k,j}(x_k u_j + x_j u_k); \beta_{k,j}(x_k u_j - x_j u_k)\}$ will be referred to as the normalized symmetric and skew-symmetric elements, respectively, of the bilinear form $\{\Pi_i(\underline{x}^{[1]}, \underline{u}^{[1]})\}$, while coefficients $\{d_{i,l}; d^*_{i,l}\}$ will be referred to as the weights on these elements.
2. Clearly, $k = m; j = n \Rightarrow l_{n,m} = \dfrac{m(2n-m+1)}{2}$, that is, in equation (3-5), the number of terms in the $sbf = l_{n,m}$, while the number of (non-zero) terms in the $dbf = l_{n,m} - m = \dfrac{m(2n-m-1)}{2}$. Note that $d^*_{i,l} = \dfrac{\alpha_{k,j} q_{k,j}^{[i]}}{\beta_{k,j}} = 0 \quad for \quad j = k$.
3. The factor $\dfrac{1}{\sqrt{2}}$ has been introduced in equation (3-5) in order to account for "normalization" of the bilinear state-input vector (defined below) similar to the normalization of the quadratic state vector considered in section 2.

3.1. Definition

The *"sum bilinear state-input vector" (sbsiv)* will be defined as the vector $\underline{\pi}^{[2]}(\underline{x}^{[1]}, \underline{u}^{[1]})$ whose elements are the (un-weighted) normalized elements of the symmetric bilinear form of equation (3-5), that is:

$$\underline{\pi}^{[2]}(\underline{x}^{[1]}, \underline{u}^{[1]}) = \left\{ \pi_l^{[2]}(\underline{x}^{[1]}, \underline{u}^{[1]}); l = 1, 2, \ldots, \frac{m(2n-m+1)}{2} \right\} \tag{3-6}$$

Where:

$$\pi_l^{[2]}(\underline{x}^{[1]}, \underline{u}^{[1]}) = \beta_{k,j}(x_k u_j + x_j u_k)$$

$$l = (k-1)\left(n - \frac{k}{2}\right) + j$$

3.2. Definition

The *"difference bilinear state-input vector" (dbsiv)* will be defined as the vector $\underline{\pi}^{[2]*}(\underline{x}^{[1]}, \underline{u}^{[1]})$ whose elements are the (un-weighted) normalized elements of the skew-symmetric bilinear form of equation (3-5), that is:

$$\underline{\pi}^{[2]*}(\underline{x}^{[1]}, \underline{u}^{[1]}) = \left\{ \pi_l^{[2]*}(\underline{x}^{[1]}, \underline{u}^{[1]}); l = 1, 2, \ldots, \frac{m(2n-m-1)}{2} \right\} \tag{3-7}$$

Where:

$$\pi_l^{[2]*}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\beta_{k,j}\left(x_k u_j - x_j u_k\right)$$

$$l=(k-1)\left(n-\frac{k}{2}\right)+j;\quad j\neq k$$

3.3. Definition

The *"bilinear state-input vector function" (bsivf)* (or the bilinear forcing function) will be defined as the vector $\underline{\Pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)$ whose elements are the (weighted) normalized elements of the bilinear form of equation (3-5), that is:

$$\underline{\Pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\left\{\Pi_i^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right);i=1,2,\ldots,n\right\} \tag{3-8}$$

Where:

$$\Pi_i\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\sum_{k=1}^{m}\sum_{j=k}^{n}\beta_{k,j}d_{k,l}\left(x_k u_j + x_j u_k\right)+\sum_{k=1}^{m}\sum_{j=k}^{n}\beta_{k,j}d^*_{k,l}\left(x_k u_j - x_j u_k\right)$$

$$=\sum_{k=1}^{m}\sum_{j=k}^{n}d_{k,l}\pi_l^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)+\sum_{k=1}^{m}\sum_{j=k}^{n}d^*_{k,l}\pi_l^{[2]*}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)$$

$$l=(k-1)\left(n-\frac{k}{2}\right)+j$$

Remarks:

1. The significance of the symmetric and skew symmetric parts will become clearer later when we consider the basis matrices that define these *bsivs*.

3.4. Example

In order to explore certain useful properties of the bilinear state-input vector (*bsiv*) or the forcing function $\underline{\Pi}^{[2]}(.,.)$, we consider equation (3-5) for n=3, m=2:

$$\Pi_1^{[2]} = d_{1,1}(x_1u_1) + d_{1,2}\frac{1}{\sqrt{2}}(x_1u_2 + x_2u_1) + d_{1,3}\frac{1}{\sqrt{2}}(x_3u_1) + d_{1,4}(x_2u_2) + d_{1,5}\frac{1}{\sqrt{2}}(x_3u_2)$$
$$+ d_{1,2}^*\frac{1}{\sqrt{2}}(x_1u_2 - x_2u_1) + d_{1,3}^*\frac{1}{\sqrt{2}}(-x_3u_1) + d_{1,5}^*\frac{1}{\sqrt{2}}(-x_3u_2)$$

$$\Pi_2^{[2]} = d_{2,1}(x_1u_1) + d_{2,2}\frac{1}{\sqrt{2}}(x_1u_2 + x_2u_1) + d_{2,3}\frac{1}{\sqrt{2}}(x_3u_1) + d_{2,4}(x_2u_2) + d_{2,5}\frac{1}{\sqrt{2}}(x_3u_2)$$
$$+ d_{2,2}^*\frac{1}{\sqrt{2}}(x_1u_2 - x_2u_1) + d_{2,3}^*\frac{1}{\sqrt{2}}(-x_3u_1) + d_{2,5}^*\frac{1}{\sqrt{2}}(-x_3u_2) \quad (3\text{-}9)$$
$$\Pi_3^{[2]} = d_{3,1}(x_1u_1) + d_{3,2}\frac{1}{\sqrt{2}}(x_1u_2 + x_2u_1) + d_{3,3}\frac{1}{\sqrt{2}}(x_3u_1) + d_{3,4}(x_2u_2) + d_{3,5}\frac{1}{\sqrt{2}}(x_3u_2)$$
$$+ d_{3,2}^*\frac{1}{\sqrt{2}}(x_1u_2 - x_2u_1) + d_{3,3}^*\frac{1}{\sqrt{2}}(-x_3u_1) + d_{3,5}^*\frac{1}{\sqrt{2}}(-x_3u_2)$$

These equations may be written in matrix notation as:

$$\Pi_1^{[2]} = [d_{1,1} \quad d_{1,2} \quad d_{1,3} \quad d_{1,4} \quad d_{1,5}]\underline{\pi}^{[2]}(.,.) + [d_{1,2}^* \quad d_{1,3}^* \quad d_{1,5}^*]\underline{\pi}^{[2]^*}(.,.)$$

$$\Pi_2^{[2]} = [d_{2,1} \quad d_{2,2} \quad d_{2,3} \quad d_{2,4} \quad d_{2,5}]\underline{\pi}^{[2]}(.,.) + [d_{2,2}^* \quad d_{2,3}^* \quad d_{2,5}^*]\underline{\pi}^{[2]^*}(.,.) \quad (3\text{-}10)$$

$$\Pi_3^{[2]} = [d_{3,1} \quad d_{3,2} \quad d_{3,3} \quad d_{3,4} \quad d_{3,5}]\underline{\pi}^{[2]}(.,.) + [d_{3,2}^* \quad d_{3,3}^* \quad d_{3,5}^*]\underline{\pi}^{[2]^*}(.,.)$$

or alternatively as:

$$\underline{\Pi}^{[2]}(\underline{x}^{[1]}, \underline{u}^{[1]}) = [D^{[2]}]\underline{\pi}^{[2]}(\underline{x}^{[1]}, \underline{u}^{[1]}) + [D^{[2]^*}]\underline{\pi}^{[2]^*}(\underline{x}^{[1]}, \underline{u}^{[1]}) \quad (3\text{-}11)$$

Where:

$[D^{[2]}] = \{d_{j,l} : j = 1,2,3; l = 1,2,\ldots,5\}$: is a $[3 \times 5]$ sum bilinear state-input (*sbsi*) coefficient matrix

$[D^{[2]^*}] = \{d_{j,l}^* : j = 1,2,3; l = 2,3,5\}$: is a $[3 \times 3]$ difference bilinear state-input (*dbsi*) coefficient matrix.

Remarks:

1. Note that $\underline{\pi}^{[2]}(\underline{x}^{[1]}, \underline{u}^{[1]})^T$ is a $[5 \times 1]$ *sbsiv*:

$$\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\left[(x_1u_1)\quad \frac{1}{\sqrt{2}}(x_1u_2+x_2u_1)\quad \frac{1}{\sqrt{2}}(x_3u_1)\quad (x_2u_2)\quad \frac{1}{\sqrt{2}}(x_3u_2)\right]^T$$

And that $\underline{\pi}^{[2]*}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)^T$ is a $[3\times 1]$ *dbsiv*:

$$\underline{\pi}^{[2]*}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\left[\frac{1}{\sqrt{2}}(x_1u_2-x_2u_1)\quad \frac{1}{\sqrt{2}}(-x_3u_1)\quad \frac{1}{\sqrt{2}}(-x_1u_2)\right]^T$$

Also, note that matrix $\left\lfloor D^{[2]*}\right\rfloor$ includes only the non-zero terms of $\{d^*_{j,l}\}$.

2. In the above example, the *sbsiv* $\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)$ may be regarded as a vector spanning the space R^5 whose elements are the terms of a bilinear polynomial in $\left\{\underline{x}^{[1]},\underline{u}^{[1]}\right\}$ arranged in lexicographic order.

3. Similarly, the *dbsiv* $\underline{\pi}^{[2]*}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)$ may also be regarded as a vector but spanning the space R^3 whose elements are the terms of a bilinear polynomial in $\left\{\underline{x}^{[1]},\underline{u}^{[1]}\right\}$ arranged in lexicographic order. However, (ignoring those terms that are always identically zero) there are two fewer terms involved in the case of *dbsiv*.

4. The inclusion of the factor $\frac{1}{\sqrt{2}}$ caters for normalization as was done for the *qsv* in the previous section. The metric property of the *sbsiv* was considered in section (2.4.4) where it was shown that for n=m:

$$\left\|\underline{\pi}^{[2]}\left(\underline{u}^{[1]},\underline{x}^{[1]}\right)\right\|\equiv\left\|\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)\right\|=\frac{1}{2}\left[\left(\underline{x}^{[1]T}\underline{u}^{[1]}\right)^2+\left\|\underline{x}^{[1]}\right\|^2\left\|\underline{u}^{[1]}\right\|^2\right] \tag{3-12}$$

For the above example n=3, m=2, it can easily be verified that:

$$\left\|\underline{\pi}^{[2]}\left(\underline{u}^{[1]},\underline{x}^{[1]}\right)\right\|\equiv\left\|\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)\right\|=\frac{1}{2}\left[\left(\underline{x}^{[1]T}\left[I_l\right]\underline{u}^{[1]}\right)^2+\left\|\underline{x}^{[1]}\right\|^2\left\|\underline{u}^{[1]}\right\|^2\right] \tag{3-13}$$

and

$$\left\|\underline{\pi}^{[2]*}\left(\underline{u}^{[1]},\underline{x}^{[1]}\right)\right\|\equiv\left\|\underline{\pi}^{[2]*}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)\right\|=\frac{1}{2}\left[\left(\underline{x}^{[1]T}\left[-I_l\right]\underline{u}^{[1]}\right)^2+\left\|\underline{x}^{[1]}\right\|^2\left\|\underline{u}^{[1]}\right\|^2\right] \tag{3-14}$$

Where:

$$\left\|\underline{\pi}^{[2]}\left(\underline{u}^{[1]},\underline{x}^{[1]}\right)\right\|=\left[\sum_{j=1}^{5}\left\{\pi_j^{[2]}\left(\underline{u}^{[1]},\underline{x}^{[1]}\right)\right\}^2\right]^{\frac{1}{2}}$$

$$\left\| \underline{\pi}^{[2]*}\left(\underline{u}^{[1]}, \underline{x}^{[1]}\right) \right\| = \left[\sum_{j=1}^{3} \left\{ \pi_j^{[2]*}\left(\underline{u}^{[1]}, \underline{x}^{[1]}\right) \right\}^2 \right]^{\frac{1}{2}}$$

$$[I_1] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

5. We observe that a bilinear polynomial (the "sum" bilinear polynomial) of the form:

$$p\left(\underline{x}^{[1]}, \underline{u}^{[1]}\right) = (x_1 u_1) + \frac{1}{\sqrt{2}}(x_1 u_2 + x_2 u_1) + \frac{1}{\sqrt{2}}(x_3 u_1) + (x_2 u_2) + \frac{1}{\sqrt{2}}(x_3 u_2)$$

may also be written as:

$$\begin{aligned} p\left(\underline{x}^{[1]}, \underline{u}^{[1]}\right) &= \left(\underline{x}^{[1]}\right)^T [F_{1,1}]\left(\underline{u}^{[1]}\right) + \left(\underline{x}^{[1]}\right)^T [F_{1,2}]\left(\underline{u}^{[1]}\right) + \left(\underline{x}^{[1]}\right)^T [F_{1,3}]\left(\underline{u}^{[1]}\right) + \left(\underline{x}^{[1]}\right)^T [F_{2,2}]\left(\underline{u}^{[1]}\right) \\ &\quad + \left(\underline{x}^{[1]}\right)^T [F_{2,3}]\left(\underline{u}^{[1]}\right) = \left(\underline{x}^{[1]}\right)^T \left\{ \sum_{j=1}^{2} \sum_{k=j}^{3} [F_{j,k}] \right\} \left(\underline{u}^{[1]}\right) \end{aligned} \tag{3-15}$$

Where:

$$[F_{1,1}] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}; \ [F_{1,2}] = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}; \ [F_{1,3}] = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}; \ [F_{2,2}] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}; \ [F_{2,3}] = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

It will be noted that if both n, m=3 then the matrices $\lfloor F_{j,k} \rfloor$ are symmetrical as in the case of the quadratic polynomial considered in section 2. Also, these matrices are similar to the basis matrices considered for the case of quadratic scalar functions and serve the same purpose. That is, they enable us to form a general symmetric bilinear scalar function $\overline{p}\left(\underline{x}^{[1]}, \underline{u}^{[1]}\right)$ in systematic way, as follows:

$$\begin{aligned} &\overline{p}\left(\underline{x}^{[1]}, \underline{u}^{[1]}\right) = \left(\underline{x}^{[1]}\right)^T \left\{ \sum_{j=1}^{2} \sum_{k=j}^{3} \lambda_l [F_{j,k}] \right\} \left(\underline{u}^{[1]}\right); \quad \lambda_l = a\ scalar \\ &l = (k-1)\left(n - \frac{k}{2}\right) + j \end{aligned} \tag{3-16}$$

6. Similarly, if we consider a bilinear polynomial (the "difference" bilinear polynomial) of the form:

$$p^*\left(\underline{x}^{[1]}, \underline{u}^{[1]}\right) = \frac{1}{\sqrt{2}}(x_1 u_2 - x_2 u_1) + \frac{1}{\sqrt{2}}(-x_3 u_1) + \frac{1}{\sqrt{2}}(-x_3 u_2)$$

then, this may also be written as:

$$\begin{aligned} p^*\left(\underline{x}^{[1]},\underline{u}^{[1]}\right) &= \left(\underline{x}^{[1]}\right)^T\left[F^*_{1,2}\right]\left(\underline{u}^{[1]}\right)+\left(\underline{x}^{[1]}\right)^T\left[F^*_{1,3}\right]\left(\underline{u}^{[1]}\right)+\left(\underline{x}^{[1]}\right)^T\left[F^*_{2,3}\right]\left(\underline{u}^{[1]}\right) \\ &= \left(\underline{x}^{[1]}\right)^T\left\{\sum_{j=1}^{2}\sum_{k=j}^{3}\left[F^*_{j,k}\right]\right\}\left(\underline{u}^{[1]}\right) \end{aligned} \tag{3-17}$$

Where:

$$\left[F^*_{1,2}\right]=\frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1\\ -1 & 0\\ 0 & 0\end{bmatrix};\ \left[F^*_{1,3}\right]=\frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0\\ 0 & 0\\ -1 & 0\end{bmatrix};\ \left[F^*_{2,3}\right]=\frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & -1\end{bmatrix};\ \left[F^*_{j,j}\right]=[0]$$

It will be noted that if n, m=3 then the matrices $\left[F^*_{j,k}\right]$ would be skew-symmetrical. These matrices are similar to the basis matrices considered earlier for the case of the "sum" bilinear polynomial and serve the same purpose. That is, they enable us to form a general skew-symmetric bilinear scalar function $\overline{p}^*\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)$ in systematic way, as follows:

$$\begin{aligned} &\overline{p}^*\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\left(\underline{x}^{[1]}\right)^T\left\{\sum_{j=1}^{2}\sum_{k=j}^{3}\lambda^*_l\left[F_{j,k}{}^*\right]\right\}\left(\underline{u}^{[1]}\right);\quad \lambda^*=a\ scalar \\ &l=(k-1)\left(n-\frac{k}{2}\right)+j;\quad \left[F^*_{j,j}\right]=[0] \end{aligned} \tag{3-18}$$

7. As discussed in section 2, the basis matrices may be used to establish a relationship between an *sbsiv* $\pi_i^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)$ and the corresponding linear state vector $\underline{x}^{[1]}$ and linear input vector $\underline{u}^{[1]}$ as follows:
Now:

$$\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\begin{bmatrix}(x_1u_1)\\ \frac{1}{\sqrt{2}}(x_1u_2+x_2u_1)\\ \frac{1}{\sqrt{2}}(x_3u_1)\\ (x_2u_2)\\ \frac{1}{\sqrt{2}}(x_3u_2)\end{bmatrix}=\begin{bmatrix}\left(\underline{x}^{[1]}\right)^T\left[F_{1,1}\right]\left(\underline{u}^{[1]}\right)\\ \left(\underline{x}^{[1]}\right)^T\left[F_{1,2}\right]\left(\underline{u}^{[1]}\right)\\ \left(\underline{x}^{[1]}\right)^T\left[F_{1,3}\right]\left(\underline{u}^{[1]}\right)\\ \left(\underline{x}^{[1]}\right)^T\left[F_{2,2}\right]\left(\underline{u}^{[1]}\right)\\ \left(\underline{x}^{[1]}\right)^T\left[F_{2,3}\right]\left(\underline{u}^{[1]}\right)\end{bmatrix}=\begin{bmatrix}\left(\underline{x}^{[1]}\right)^T\left[F_{1,1}\right]\\ \left(\underline{x}^{[1]}\right)^T\left[F_{1,2}\right]\\ \left(\underline{x}^{[1]}\right)^T\left[F_{1,3}\right]\\ \left(\underline{x}^{[1]}\right)^T\left[F_{2,2}\right]\\ \left(\underline{x}^{[1]}\right)^T\left[F_{2,3}\right]\end{bmatrix}\left[\underline{u}^{[1]}\right] \tag{3-19}$$

$$=\begin{bmatrix} x_1 & 0 \\ \frac{x_2}{\sqrt{2}} & \frac{x_1}{\sqrt{2}} \\ \frac{x_3}{\sqrt{2}} & 0 \\ 0 & x_2 \\ 0 & \frac{x_3}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \left[\overline{X}{}^{[2]}_{[1]}\right]\underline{u}^{[1]} = \begin{bmatrix} \left(\underline{u}^{[1]}\right)^T [F_{1,1}]^T \\ \left(\underline{u}^{[1]}\right)^T [F_{1,2}]^T \\ \left(\underline{u}^{[1]}\right)^T [F_{1,3}]^T \\ \left(\underline{u}^{[1]}\right)^T [F_{2,2}]^T \\ \left(\underline{u}^{[1]}\right)^T [F_{2,3}]^T \end{bmatrix}\left[\underline{x}^{[1]}\right] = \begin{bmatrix} u_1 & 0 & 0 \\ \frac{u_2}{\sqrt{2}} & \frac{u_1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{u_1}{\sqrt{2}} \\ 0 & u_2 & 0 \\ 0 & 0 & \frac{u_2}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left[\overline{U}{}^{[2]}_{[1]}\right]\underline{x}^{[1]} \tag{3-20}$$

Where:

$$\left[\overline{X}{}^{[2]}_{[1]}\right] = \begin{bmatrix} x_1 & 0 \\ \frac{x_2}{\sqrt{2}} & \frac{x_1}{\sqrt{2}} \\ \frac{x_3}{\sqrt{2}} & 0 \\ 0 & x_2 \\ 0 & \frac{x_3}{\sqrt{2}} \end{bmatrix}; \text{and } \left[\overline{U}{}^{[2]}_{[1]}\right] = \begin{bmatrix} u_1 & 0 & 0 \\ \frac{u_2}{\sqrt{2}} & \frac{u_1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{u_1}{\sqrt{2}} \\ 0 & u_2 & 0 \\ 0 & 0 & \frac{u_2}{\sqrt{2}} \end{bmatrix}$$

8. The above result can be generalized to the case of $\underline{x}^{[1]} \in R^n$, $\underline{u}^{[1]} \in R^m$. Matrices $\left[\overline{X}{}^{[2]}_{[1]}\right] \in R^{l_1 x m}$ and $\left[\overline{U}{}^{[2]}_{[1]}\right] \in R^{l_1 x n}$, $l_1 = \frac{m(2n-m+1)}{2}$, will be referred to as the sum bilinear generator matrices (*sbgms*) that map respectively $\underline{u}^{[1]} \in R^m$ and $\underline{x}^{[1]} \in R^n$ onto $\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right) \in R^{l_1}$; That is:

$$\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right) = \left[\overline{X}{}^{[2]}_{[1]}\right]\underline{u}^{[1]} = \left[\overline{U}{}^{[2]}_{[1]}\right]\underline{x}^{[1]} \tag{3-21}$$

It can be shown, that provided not all the elements of $\underline{u}^{[1]}$and $\underline{x}^{[1]}$ are identically zero, then there exist the *inverse sbgms* (*isbgms*) $\left[\overline{X}{}^{[1]}_{[2]}\right] = \left[\overline{X}{}^{[2]}_{[1]}\right]^{-1}$, $\left[\overline{U}{}^{[1]}_{[2]}\right] = \left[\overline{U}{}^{[2]}_{[1]}\right]^{-1}$ such that:

$$\underline{u}^{[1]} = \left[\overline{X}{}^{[1]}_{[2]}\right]\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right); \text{and } \underline{x}^{[1]} = \left[\overline{U}{}^{[1]}_{[2]}\right]\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right) \tag{3-22}$$

Note the changed super- and sub-scripts in the inverse generator matrices.

Where:

$\left[\overline{X}{}^{[1]}_{[2]}\right] = \left\{\left[\overline{X}{}^{[2]}_{[1]}\right]^T\left[\overline{X}{}^{[2]}_{[1]}\right]\right\}^{-1}\left[\overline{X}{}^{[2]}_{[1]}\right]^T$: is an $[m \times l_1]$ inverse sum bilinear generator matrix for $\underline{u}^{[1]}$.

$\left[\overline{U}_{[2]}^{[1]}\right]=\left\{\left[\overline{U}_{[1]}^{[2]}\right]^T\left[\overline{U}_{[1]}^{[2]}\right]\right\}^{-1}\left[\overline{U}_{[1]}^{[2]}\right]^T$: is an $[n\times l_1]$ inverse sum bilinear generator matrix for $\underline{x}^{[1]}$.

9. For our particular example, it can be shown that:

$$\left\{\left[\overline{X}_{[1]}^{[2]}\right]^T\left[\overline{X}_{[1]}^{[2]}\right]\right\}=\frac{1}{2}\begin{bmatrix}\left(2x_1^2+x_2^2+x_3^2\right) & x_1x_2\\ x_1x_2 & \left(x_1^2+2x_2^2+x_3^2\right)\end{bmatrix}$$

$$\left\{\left[\overline{X}_{[1]}^{[2]}\right]^T\left[\overline{X}_{[1]}^{[2]}\right]\right\}^{-1}=\frac{2}{\Delta}\begin{bmatrix}\left(x_1^2+2x_2^2+x_3^2\right) & -x_1x_2\\ -x_1x_2 & \left(2x_1^2+x_2^2+x_3^2\right)\end{bmatrix}$$

Where:

$$\Delta=\left[\left(x_1^2+x_2^2+x_3^2\right)\left(2x_1^2+2x_2^2+x_3^2\right)\right]$$

And:

$$\left[\overline{X}_{[2]}^{[1]}\right]=$$

$$\frac{2}{\Delta}\begin{bmatrix}x_1\left(x_1^2+2x_2^2+x_3^2\right) & \frac{x_2\left(2x_2^2+x_3^2\right)}{\sqrt{2}} & \frac{x_3\left(x_1^2+2x_2^2+x_3^2\right)}{\sqrt{2}} & \left(-x_1x_2^2\right) & \frac{\left(-x_1x_2x_3\right)}{\sqrt{2}}\\ \left(-x_1^2x_2\right) & \frac{x_1\left(2x_1^2+x_3^2\right)}{\sqrt{2}} & \frac{\left(-x_1x_2x_3\right)}{\sqrt{2}} & x_2\left(2x_1^2+x_2^2+x_3^2\right) & \frac{x_3\left(2x_1^2+x_2^2+x_3^2\right)}{\sqrt{2}}\end{bmatrix}$$

(3-23)

10. Similarly it can be shown that:

$$\left\{\left[\overline{U}_{[1]}^{[2]}\right]^T\left[\overline{U}_{[1]}^{[2]}\right]\right\}=\begin{bmatrix}\left(u_1^2+\frac{u_2^2}{2}\right) & \left(\frac{u_1u_2}{2}\right) & 0\\ \left(\frac{u_1u_2}{2}\right) & \left(\frac{u_1^2}{2}+u_2^2\right) & 0\\ 0 & 0 & \left(u_1^2+u_2^2\right)\end{bmatrix}$$

$$\left\{\left[\overline{U}_{[1]}^{[2]}\right]^T\left[\overline{U}_{[1]}^{[2]}\right]\right\}^{-1}=\frac{1}{\Delta}\begin{bmatrix} 2\left(\frac{u_1^2}{2}+u_2^2\right) & (-u_1u_2) & 0 \\ (-u_1u_2) & 2\left(u_1^2+\frac{u_2^2}{2}\right) & 0 \\ 0 & 0 & 2\left(u_1^2+u_2^2\right) \end{bmatrix}$$

Where:

$\Delta=\left(u_1^2+u_2^2\right)^2$

And:

$$\left[\overline{U}_{[2]}^{[1]}\right]=\frac{1}{\Delta}\begin{bmatrix} 2u_1\left(\frac{u_1^2}{2}+u_2^2\right) & \frac{2u_2^3}{\sqrt{2}} & 0 & \left(-u_1u_2^2\right) & 0 \\ \left(-u_1u_2^2\right) & \frac{2u_2^3}{\sqrt{2}} & 0 & 2u_2\left(u_1^2+\frac{u_2^2}{2}\right) & 0 \\ 0 & 0 & \frac{2u_1\left(u_1^2+u_1^2\right)}{\sqrt{2}} & 0 & \frac{2u_1\left(u_1^2+u_1^2\right)}{\sqrt{2}} \end{bmatrix}$$

(3-24)

11. Consideration, analogous to those discussed in remarks 7-10 above, also apply for the case of *dbsiv*; the results are quoted below:

$$\underline{\pi}^{[2]*}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\begin{bmatrix} \frac{1}{\sqrt{2}}(x_1u_2-x_2u_1) \\ \frac{1}{\sqrt{2}}(-x_3u_1) \\ \frac{1}{\sqrt{2}}(-x_3u_2) \end{bmatrix}=\begin{bmatrix} \left(\underline{x}^{[1]}\right)^T\left[F_{1,2}^{\ *}\right]\left(\underline{u}^{[1]}\right) \\ \left(\underline{x}^{[1]}\right)^T\left[F_{1,3}^{\ *}\right]\left(\underline{u}^{[1]}\right) \\ \left(\underline{x}^{[1]}\right)^T\left[F_{2,3}^{\ *}\right]\left(\underline{u}^{[1]}\right) \end{bmatrix}=\begin{bmatrix} \left(\underline{x}^{[1]}\right)^T\left[F_{1,2}^{\ *}\right] \\ \left(\underline{x}^{[1]}\right)^T\left[F_{1,3}^{\ *}\right] \\ \left(\underline{x}^{[1]}\right)^T\left[F_{2,3}^{\ *}\right] \end{bmatrix}\left[\underline{u}^{[1]}\right]$$

$$=\begin{bmatrix} -\frac{x_2}{\sqrt{2}} & \frac{x_1}{\sqrt{2}} \\ -\frac{x_3}{\sqrt{2}} & 0 \\ 0 & -\frac{x_3}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}=\left[\overline{X}_{[1]}^{[2]*}\right]\underline{u}^{[1]} \qquad (3\text{-}25)$$

$$= \begin{bmatrix} \left(\underline{u}^{[1]}\right)^T \left[F_{1,2}{}^*\right]^T \\ \left(\underline{u}^{[1]}\right)^T \left[F_{1,3}{}^*\right]^T \\ \left(\underline{u}^{[1]}\right)^T \left[F_{2,3}{}^*\right]^T \end{bmatrix} \left[\underline{x}^{[1]}\right] = \begin{bmatrix} \frac{u_2}{\sqrt{2}} & -\frac{u_1}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{u_1}{\sqrt{2}} \\ 0 & 0 & -\frac{u_2}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left[\overline{U}_{[1]}^{[2]*}\right] \underline{x}^{[1]} \qquad (3\text{-}26)$$

Where:

$$\left[\overline{X}_{[1]}^{[2]*}\right] = \begin{bmatrix} -\frac{x_2}{\sqrt{2}} & \frac{x_1}{\sqrt{2}} \\ -\frac{x_3}{\sqrt{2}} & 0 \\ 0 & -\frac{x_3}{\sqrt{2}} \end{bmatrix} \text{; and } \left[\overline{U}_{[1]}^{[2]*}\right] = \begin{bmatrix} \frac{u_2}{\sqrt{2}} & -\frac{u_1}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{u_1}{\sqrt{2}} \\ 0 & 0 & -\frac{u_2}{\sqrt{2}} \end{bmatrix}$$

12. The above result can be generalized to the case of $\underline{x}^{[1]} \in R^n$, $\underline{u}^{[1]} \in R^m$. Matrices $\left[\overline{X}_{[1]}^{[2]*}\right] \in R^{l_2 x m}$ and $\left[\overline{U}_{[1]}^{[2]*}\right] \in R^{l_2 x n}$ will be referred to as the difference bilinear generator matrices (*dbgms*) that map respectively $\underline{u}^{[1]} \in R^m$ and $\underline{x}^{[1]} \in R^n$, $l_2 = \frac{m(2n-m-1)}{2}$, onto $\underline{\pi}^{[2]*}\left(\underline{x}^{[1]}, \underline{u}^{[1]}\right) \in R^{l_2}$, That is:

$$\underline{\pi}^{[2]*}\left(\underline{x}^{[1]}, \underline{u}^{[1]}\right) = \left[\overline{X}_{[1]}^{[2]*}\right] \underline{u}^{[1]} = \left[\overline{U}_{[1]}^{[2]*}\right] \underline{x}^{[1]} \qquad (3\text{-}27)$$

13. Unlike the previous case, the existence of an inverse generator matrix is not guaranteed. In fact, it can be shown that an *inverse dbgm* (*idbgm*) $\left[\overline{U}_{[2]}^{[1]*}\right]$ as defined by the relation:

$$\left[\overline{U}_{[2]}^{[1]*}\right] = \left\{\left[\overline{U}_{[1]}^{[2]*}\right]^T \left[\overline{U}_{[1]}^{[2]*}\right]\right\}^{-1} \left[\overline{U}_{[1]}^{[2]*}\right]^T$$: does not exist, since the inversion is not defined.

However, it may be shown, that provided that none of the elements of $\underline{x}^{[1]}$ are identically zero, then there exist an *inverse dbgm* (*isbgm*) $\left[\overline{X}_{[2]}^{[1]*}\right]$ such that:

$$\underline{u}^{[1]} = \left[\overline{X}_{[2]}^{[1]*}\right] \underline{\pi}^{[2]*}\left(\underline{x}^{[1]}, \underline{u}^{[1]}\right)$$

Note the changed super- and sub-scripts in the inverse generator matrices.

Where:

$\left[\overline{X}_{[2]}^{[1]*}\right]=\left\{\left[\overline{X}_{[1]}^{[2]*}\right]^{T}\left[\overline{X}_{[1]}^{[2]*}\right]\right\}^{-1}\left[\overline{X}_{[1]}^{[2]*}\right]^{T}$: is an $[m\times l_2]$ inverse sum bilinear generator matrix for $\underline{u}^{[1]}$.

For the current example:

$$\left[\overline{X}_{[2]}^{[1]*}\right]=\frac{2}{\Delta}\begin{bmatrix}-x_2x_3^{\,2} & -x_3\left(x_1^{\,2}+x_3^{\,2}\right) & -x_1x_2x_3 \\ x_1x_3^{\,2} & -x_1x_2x_3 & -x_3\left(x_2^{\,2}+x_3^{\,2}\right)\end{bmatrix} \qquad (3\text{-}28)$$

Where:

$\Delta=x_3^{\,2}\left(x_1^{\,2}+x_2^{\,2}+x_3^{\,2}\right)$

3.5. Generalised Bilinear State-Input Generator Matrices

For a state vector $\underline{x}^{[1]}\in R^n$ and an input vector $\underline{u}^{[1]}\in R^m$, the *sbgms* were defined in section 2, equations (B1-2),(B1-3). The *sbgms* and *dbgms* matrices may also be constructed utilizing the basis matrices $\lfloor F_{j,k}\rfloor$, $\lfloor F_{j,k}^{\,*}\rfloor$. Following the procedure given in equations (3-19), (3-20) of the example, an algorithm for constructing the basis matrices $\lfloor F_{j,k}\rfloor$ for a general case, may be defined as follows:

$$\left\{F_{k,j}\right\}_{r,s}=\begin{cases} for \quad j=k & \begin{cases} 1 & for \quad r=s \\ 0 & otherwise\end{cases} \\ for \quad j\neq k & \begin{cases} \dfrac{1}{\sqrt{2}} & for \quad r=k,s=j \;and\; r=j,s=k \\ 0 & otherwise\end{cases}\end{cases} \qquad (3\text{-}29)$$

$j=1,2,\ldots,m;k=j,j+1,\ldots,n$

The basis matrices for a general n-state, m-input symmetric bilinear vector system are given below:

$$\left[F_{1,1}\right]=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix};\ \left[F_{1,2}\right]=\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix};\ \left[F_{1,3}\right]=\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix};\ldots;$$

$$\left[F_{1,m}\right]=\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix};\ldots;\ \left[F_{1,n}\right]=\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix};\ \left[F_{2,2}\right]=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\left[F_{2,3}\right]=\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix};\ldots;\ \left[F_{2,n}\right]=\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix};\ldots;\left[F_{m,n}\right]=\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Similarly, the *dbsigm* may be constructed using the basis matrices $\left[F_{j,k}^{*}\right]$, following the procedure given in equations (3-25), (3-26) of the example, an algorithm for constructing these basis matrices may be defined as follows:

$$\left\{F_{k,j}^{*}\right\}_{r,s}=\begin{cases} \dfrac{1}{\sqrt{2}} & for \quad r=k,s=j \\ \dfrac{-1}{\sqrt{2}} & for \quad r=j,s=k \\ 0 \quad otherwise & \end{cases} \tag{3-30}$$

$$j=1,2,\ldots,m;k=j,j+1,\ldots,n; \quad and\ j\neq k$$

The basis matrices for a general n-state, m-input skew symmetric bilinear vector system are given below:

$$\left[F^*_{1,2}\right]=\frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0&\dots&0\\-1&0&0&\dots&0\\0&0&0&\dots&0\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\0&0&0&\dots&0\end{bmatrix};\ \left[F^*_{1,3}\right]=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&1&\dots&0\\0&0&0&\dots&0\\-1&0&0&\dots&0\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\0&0&0&\dots&0\end{bmatrix};\dots;\ \left[F^*_{1,m}\right]=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&\dots&1\\0&0&0&\dots&0\\0&0&0&\dots&0\\\dots&\dots&\dots&\dots&\dots\\-1&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\0&0&0&\dots&0\end{bmatrix};\dots;$$

$$\left[F^*_{1,n}\right]=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&\dots&0\\0&0&0&\dots&0\\0&0&0&\dots&0\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\-1&0&0&\dots&0\end{bmatrix};\ \left[F^*_{2,3}\right]=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&\dots&0\\0&0&1&\dots&0\\0&-1&0&\dots&0\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\0&0&0&\dots&0\end{bmatrix};\dots;\ \left[F^*_{2,n}\right]=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&\dots&0\\0&0&0&\dots&0\\0&0&0&\dots&0\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\0&-1&0&\dots&0\end{bmatrix};\dots;$$

$$\left[F^*_{m,m+1}\right]=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&\dots&0\\0&0&0&\dots&0\\0&0&0&\dots&0\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&0\\\dots&\dots&\dots&\dots&-1\\\dots&\dots&\dots&\dots&\dots\\0&0&0&\dots&0\end{bmatrix};\dots;\ \left[F^*_{m,n}\right]=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&\dots&0\\0&0&0&\dots&0\\0&0&0&\dots&0\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\0&0&0&\dots&-1\end{bmatrix}$$

The *dbgm* matrices $\left[\overline{X}^{[2]^*}_{[1]}\right],\left[\overline{U}^{[2]^*}_{[1]}\right]$ are given in Appendix-B, equations (B1-4), (B1-5).

3.6. Bilinear State-Input Vector Function

Finally before we leave this section, we give a state space representation of the state-input vector (or a forcing) function. Using relationships developed in this section, a bilinear n-state, m-input vector function may be written as:

$$bsiv=\underline{\Pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\left[D^{[2]}\right]\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)+\left[D^{[2]^*}\right]\underline{\pi}^{[2]^*}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right) \tag{3-31}$$

Or alternatively, in terms of the *bgms*, as:

$$\underline{\Pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\left[D^{[2]}\right]\left[\overline{X}^{[2]}_{[1]}\right]\underline{u}^{[1]}+\left[D^{[2]^*}\right]\left[\overline{X}^{[2]^*}_{[1]}\right]\underline{u}^{[1]}=\left\{\left[D^{[2]}\right]\left[\overline{X}^{[2]}_{[1]}\right]+\left[D^{[2]^*}\right]\left[\overline{X}^{[2]^*}_{[1]}\right]\right\}\underline{u}^{[1]}$$

$$\underline{\Pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\left[D^{[2]}\right]\left[\overline{U}^{[2]}_{[1]}\right]\underline{x}^{[1]}+\left[D^{[2]^*}\right]\left[\overline{U}^{[2]^*}_{[1]}\right]\underline{x}^{[1]}=\left\{\left[D^{[2]}\right]\left[\overline{U}^{[2]}_{[1]}\right]+\left[D^{[2]^*}\right]\left[\overline{U}^{[2]^*}_{[1]}\right]\right\}\underline{x}^{[1]} \tag{3-32}$$

4. Forced Quadratic-Bilinear Dynamical System

We shall now consider the algebraic structure of a forced-quadratic/bilinear system where the RHS of the differential equations consists of linear and quadratic terms identical to those in the free-QDS as well as additional linear input terms and bilinear terms involving inputs (forcing functions or control inputs) and system states.

4.1 Definition

A dynamical system will be classed as a forced-quadratic/bilinear dynamical system (forced-QBDS) if the right hand side (RHS) of the differential equations describing the time evolution of states consists of (see sections 2 and 3):

a. a linear term in system state vector: $\left[A^{[1]}\right]\underline{x}^{[1]}$
b. a quadratic term in system state vector: $\left[B^{[2]}\right]\underline{x}^{[2]}$
c. a linear term in input vector: $\left[C^{[1]}\right]\underline{u}^{[1]}$
d. a bilinear term in state-input vector:
$\underline{\Pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\left[D^{[2]}\right]\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)+\left[D^{[2]^*}\right]\underline{\pi}^{[2]^*}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)$

That is, the forced-QBDS may be written as:

$$\begin{aligned}\frac{d}{dt}\underline{x}^{[1]}&=\left[A^{[1]}\right]\underline{x}^{[1]}+\left[B^{[2]}\right]\underline{x}^{[2]}+\left[C^{[1]}\right]\underline{u}^{[1]}+\underline{\Pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)\\&=\left[A^{[1]}\right]\underline{x}^{[1]}+\left[B^{[2]}\right]\underline{x}^{[2]}+\left[C^{[1]}\right]\underline{u}^{[1]}+\left[D^{[2]}\right]\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)+\left[D^{[2]^*}\right]\underline{\pi}^{[2]^*}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)\end{aligned} \tag{4-1}$$

Where:

$\left[A^{[1]}\right]=\left\{a_{i,j};i=1,2,...,n;\ j=1,2,...,n\right\}$: is an $[n\times n]$ linear state coefficient matrix

$\left[B^{[2]}\right]=\left\{b_{i,l};i=1,2,...n;l=1,2,...,\frac{n(n+1)}{2}\right\}$: is an $\left[n\ x\ \frac{n(n+1)}{2}\right]$ quadratic state coefficient matrix

$\left[C^{[1]}\right]=\left\{c_{i,k};i=1,2,...,n;k=1,2,...,m\le n\right\}$: is an $[n\times m]$ input coefficient matrix

$\left[D^{[2]}\right]=\left\{d_{j,l}:j=1,2,3,...,n;l=1,2,3,...,\frac{m(2n-m+1)}{2}\right\}$: is an $\left[n\times\frac{m(2n-m+1)}{2}\right]$ sum bilinear state-input (*sbsi*) coefficient matrix

$\left[D^{[2]^*}\right]=\left\{d^*_{j,l}:j=1,2,3,...,n;l=2,3,5,...,\frac{m(2n-m-1)}{2}\right\}$: is an $\left[n\times\frac{m(2n-m-1)}{2}\right]$ difference bilinear state-input (*dbsi*) coefficient matrix

$\underline{x}^{[1]}=\left\{x_i^{[1]};i=1,2,...,n\right\}$: is an $[n\times 1]$ linear state vector

$\underline{x}^{[2]}=\left\{x_i^{[2]};i=1,2,...,\frac{n(n+1)}{2}\right\}$: is an $\left[\frac{n(n+1)}{2}\times 1\right]$ quadratic state vector

$\underline{u}^{[1]}=\left\{u_i^{[1]};i=1,2,...,m\le n\right\}$: is an $[m\times 1]$ linear input vector

$\underline{\Pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\left\{\Pi_i^{[2]};i=1,2,\ldots,n\right\}$: is an $[n\times 1]$ bilinear state-input vector function

$\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\left\{\pi_i^{[2]};i=1,2,\ldots,\frac{m(2n-m+1)}{2}\right\}$: is an $\left[\frac{m(2n-m+1)}{2}\times 1\right]$ sum bilinear state-input (*sbsi*) vector

$\underline{\pi}^{[2]*}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)=\left\{\pi_i^{[2]*};i=1,2,\ldots,\frac{m(2n-m+1)}{2}\right\}$: is an $\left[\frac{m(2n-m-1)}{2}\times 1\right]$ difference bilinear state-input (*sbsi*) vector

Alternatively, utilising the *qgm* and *bgm* defined in sections 2 and 3, the Forced-QBDS may also be written as:

$$\frac{d}{dt}\underline{x}^{[1]}=\left\{\left[A^{[1]}\right]+\left[B^{[2]}\right]\left[X_{[1]}^{[2]}\right]\right\}\underline{x}^{[1]}+\left\{\left[C^{[1]}\right]+\left[D^{[2]}\right]\left[\overline{X}_{[1]}^{[2]}\right]+\left[D^{[2]*}\right]\left[\overline{X}_{[1]}^{[2]*}\right]\right\}\underline{u}^{[1]} \tag{4-2a}$$

Or equivalently as:

$$\frac{d}{dt}\underline{x}^{[1]}=\left\{\left[A^{[1]}\right]+\left[B^{[2]}\right]\left[X_{[1]}^{[2]}\right]+\left[D^{[2]}\right]\left[\overline{U}_{[1]}^{[2]}\right]+\left[D^{[2]*}\right]\left[\overline{U}_{[1]}^{[2]*}\right]\right\}\underline{x}^{[1]}+\left[C^{[1]}\right]\underline{u}^{[1]} \tag{4-2b}$$

The forcing function or the input vector function $\underline{u}_I=\left\{u_{I_i};i=1,2,\ldots,n\right\}$ is given by:

$$\underline{u}_I=\left[C^{[1]}\right]\underline{u}^{[1]}+\underline{\Pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)$$

$$=\left[C^{[1]}\right]\underline{u}^{[1]}+\left[D^{[2]}\right]\underline{\pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)+\left[D^{[2]*}\right]\underline{\pi}^{[2]*}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right) \tag{4-3a}$$

$$=\left\{\left[C^{[1]}\right]+\left[D^{[2]}\right]\left[\overline{X}_{[1]}^{[2]}\right]+\left[D^{[2]*}\right]\left[\overline{X}_{[1]}^{[2]*}\right]\right\}\underline{u}^{[1]} \tag{4-3b}$$

$$=\left[C^{[1]}\right]\underline{u}^{[1]}+\left\{\left[D^{[2]}\right]\left[\overline{U}_{[1]}^{[2]}\right]+\left[D^{[2]*}\right]\left[\overline{U}_{[1]}^{[2]*}\right]\right\}\underline{x}^{[1]} \tag{4-3c}$$

Where:

$\left[X_{[1]}^{[2]}\right];\left[\overline{X}_{[1]}^{[2]}\right];\left[\overline{U}_{[1]}^{[2]}\right];\left[\overline{X}_{[1]}^{[2]*}\right];\left[\overline{U}_{[1]}^{[2]*}\right]$: are the *qgm* and *bgms* defined earlier in sections 2 and 3.

4.2. Example:

For n=3 and m=2, the forcing function or the input term $\underline{u}_I=\left[C^{[1]}\right]\underline{u}^{[1]}+\underline{\Pi}^{[2]}\left(\underline{x}^{[1]},\underline{u}^{[1]}\right)$ may be written as:

$$u_{F_1} = c_{1,1}u_1 + c_{1,2}u_2 + d_{1,1}x_1u_1 + d_{1,2}\frac{1}{\sqrt{2}}(x_1u_2 + x_2u_1) + d_{1,3}\frac{1}{\sqrt{2}}(x_3u_1) + d_{1,4}x_2u_2$$

$$+ d_{1,5}\frac{1}{\sqrt{2}}(x_3u_2) + d^*_{1,2}\frac{1}{\sqrt{2}}(x_1u_2 - x_2u_1) + d^*_{1,3}\frac{1}{\sqrt{2}}(-x_3u_1) + d^*_{1,5}\frac{1}{\sqrt{2}}(-x_3u_2)$$

$$u_{F_2} = c_{2,1}u_1 + c_{2,2}u_2 + d_{2,1}x_1u_1 + d_{2,2}\frac{1}{\sqrt{2}}(x_1u_2 + x_2u_1) + d_{2,3}\frac{1}{\sqrt{2}}(x_3u_1) + d_{2,4}x_2u_2$$

$$+ d_{2,5}\frac{1}{\sqrt{2}}(x_3u_2) + d^*_{2,2}\frac{1}{\sqrt{2}}(x_1u_2 - x_2u_1) + d^*_{2,3}\frac{1}{\sqrt{2}}(-x_3u_1) + d^*_{2,5}\frac{1}{\sqrt{2}}(-x_3u_2)$$

$$u_{F_3} = c_{3,1}u_1 + c_{3,2}u_2 + d_{3,1}x_1u_1 + d_{3,2}\frac{1}{\sqrt{2}}(x_1u_2 + x_2u_1) + d_{3,3}\frac{1}{\sqrt{2}}(x_3u_1) + d_{1,4}x_2u_2$$

$$+ d_{3,5}\frac{1}{\sqrt{2}}(x_3u_2) + d^*_{3,2}\frac{1}{\sqrt{2}}(x_1u_2 - x_2u_1) + d^*_{3,3}\frac{1}{\sqrt{2}}(-x_3u_1) + d^*_{3,5}\frac{1}{\sqrt{2}}(-x_3u_2)$$

Remarks: Compare the above result with equation (3-11).

5. Quadratic-Bilinear System Output Model

Utilising the definitions given in previous sections, dynamical system output or measurement models may be derived. Following on from the results of previous sections, the system output model may be written as:

$$\underline{z}^{[1]} = \left[H^{[1]}\right]\underline{x}^{[1]} + \left[J^{[2]}\right]\underline{x}^{[2]} + \underline{\Pi}^{[2]}\left(\underline{x}^{[1]}, \underline{v}^{[1]}\right) + \left[L^{[1]}\right]\underline{v}^{[1]} \tag{5-1a}$$

$$= \left[H^{[1]}\right]\underline{x}^{[1]} + \left[J^{[2]}\right]\underline{x}^{[2]} + \left[M^{[2]}\right]\underline{\pi}^{[2]}\left(\underline{x}^{[1]}, \underline{v}^{[1]}\right) + \left[M^{[2]^*}\right]\underline{\pi}^{[2]^*}\left(\underline{x}^{[1]}, \underline{v}^{[1]}\right) + \left[L^{[1]}\right]\underline{v}^{[1]} \tag{5-1b}$$

Or using *qgm* and *bgms* as:

$$\underline{z}^{[1]} = \left\{\left[H^{[1]}\right] + \left[J^{[2]}\right]\left[X^{[2]}_{[1]}\right]\right\}\underline{x}^{[1]} + \left\{\left[L^{[1]}\right] + \left[M^{[2]}\right]\left[\overline{X}^{[2]}_{[1]}\right] + \left[M^{[2]^*}\right]\left[\overline{X}^{[2]^*}_{[1]}\right]\right\}\underline{v}^{[1]} \tag{5-2a}$$

$$\underline{z}^{[1]} = \left\{\left[H^{[1]}\right] + \left[J^{[2]}\right]\left[X^{[2]}_{[1]}\right] + \left[M^{[2]}\right]\left[\overline{V}^{[2]}_{[1]}\right] + \left[M^{[2]^*}\right]\left[\overline{V}^{[2]^*}_{[1]}\right]\right\}\underline{x}^{[1]} + \left[L^{[1]}\right]\underline{v}^{[1]} \tag{5-2b}$$

Where:

$\left[H^{[1]}\right] = \{h_{i,j}; i = 1,2,...,r; j = 1,2,...,n\}$: is an $[r \times n]$ linear state output coefficient matrix

$\left[J^{[2]}\right] = \left\{j_{i,l}; i = 1,2,...r; l = 1,2,...,\frac{n(n+1)}{2}\right\}$: is an $\left[r \times \frac{n(n+1)}{2}\right]$ quadratic state output coefficient matrix

$\left[L^{[1]}\right] = \{l_{i,k}; i = 1,2,...,r; k = 1,2,...,s \le n\}$: is an $[r \times s]$ output disturbance coefficient matrix

$\left[M^{[2]}\right]=\left\{m_{i,p}; i=1,2,...r; p=1,2,...,\frac{s(2n-s+1)}{2}\right\}$: is an $\left[r\times\frac{s(2n-s+1)}{2}\right]$ sum bilinear state-disturbance (*sbsd*) coefficient matrix

$\left[M^{[2]^*}\right]=\left\{m^*_{i,p}; i=1,2,...r; p=1,2,...,\frac{s(2n-s-1)}{2}\right\}$: is an $\left[r\times\frac{s(2n-s-1)}{2}\right]$ difference bilinear state-disturbance (*dbsd*) coefficient matrix. Note that matrices $\left[M^{[2]}\right]\left[M^{[2]^*}\right]$ are similar to $\left[D^{[2]}\right]\left[D^{[2]^*}\right]$ of the previous sections 3 and 4.

$\underline{z}^{[1]}=\left\{z_i^{[1]}; i=1,2,...,r\right\}$: is the $[r\times 1]$ output (measurement) vector

$\underline{v}^{[1]}=\left\{v_i^{[1]}; i=1,2,...,s\right\}$: is the $[s\times 1]$ output disturbance vector

$\left[\overline{X}^{[2]}_{[1]}\right];\left[\overline{X}^{[2]^*}_{[1]}\right]$: are respectively $\left[\frac{s(2n-s+1)}{2}\times s\right]$ and $\left[\frac{s(2n-s-1)}{2}\times s\right]$ *bgms* containing elements of $\underline{x}^{[1]}$ similar to those have been defined in sections 2 and 3.

$\left[\overline{V}^{[2]}_{[1]}\right];\left[\overline{V}^{[2]^*}_{[1]}\right]$: are respectively $\left[\frac{s(2n-s+1)}{2}\times n\right]$ and $\left[\frac{s(2n-s-1)}{2}\times n\right]$ *bgms* containing elements of $\underline{v}^{[1]}$ similar to those defined in sections 2 and 3.

A block diagrams for the forced-QBDS with output model is given in Figures 5.1. Figure 5.2 gives a block diagram where control input and disturbance vectors are linear.

5.1. An Example of a Second Order Matrix Riccati Differential Equation:

We consider the matrix Riccati differential equation of the type commonly encountered in synthesis of an optimal controller or a filter for a linear dynamical system [9,10]. The matrix Ricatti differential equation resulting from the estimation (filtering) problem for a second order system is as follows:

$$\frac{d}{dt}[P]=[P][A]+[A]^T[P]-[P][S][P]+[Q] \tag{5-3}$$

Where:

$[A]=\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$: is a $[2\times 2]$ linear state coefficient matrix

$[Q]=\begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix}$: is a $[2\times 2]$ (symmetric) system noise covariance matrix

$[P]=\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$: is a $[2\times 2]$ (symmetric) Riccati matrix

$[S] = \begin{bmatrix} s_{1,1} & s_{1,2} \\ s_{1,2} & s_{2,2} \end{bmatrix}$: is a $[2 \times 2]$ (symmetric) matrix

Expanding equation (3-33) in its elemental form gives us:

$$\frac{d}{dt}p_1 = 2(a_{1,1}p_1 + a_{1,2}p_2) - \left[s_{1,1}p_1^{\ 2} + 2s_{1,2}p_1p_2 + s_{2,2}p_2^{\ 2}\right] + q_1$$
$$\frac{d}{dt}p_2 = a_{2,1}p_1 + (a_{1,1} + a_{2,2})p_2 + a_{2,1}p_3 - \left[s_{1,2}p_2^{\ 2} + s_{1,1}p_1p_2 + s_{1,2}p_1p_3 + s_{2,2}p_2p_3\right] + q_2$$
$$\frac{d}{dt}p_3 = 2(a_{2,1}p_2 + a_{2,2}p_3) - \left[s_{1,1}p_2^{\ 2} + 2s_{1,2}p_2p_3 + s_{2,2}p_3^{\ 2}\right] + q_3$$

(5-4)

Equation (3-34) may be written in matrix form as:

$$\frac{d}{dt}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 2a_{1,1} & 2a_{1,2} & 0 \\ a_{2,1} & (a_{1,1} + a_{2,2}) & a_{2,1} \\ 0 & 2a_{2,1} & 2a_{2,2} \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} s_{1,1} & s_{1,2} & 0 & s_{2,2} & 0 & 0 \\ 0 & s_{1,1} & s_{1,2} & s_{1,2} & s_{2,2} & 0 \\ 0 & 0 & 0 & s_{1,1} & s_{1,2} & s_{2,2} \end{bmatrix}\begin{bmatrix} p_1^{\ 2} \\ p_1p_2 \\ p_1p_3 \\ p_2^{\ 2} \\ p_2p_3 \\ p_3^{\ 2} \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

(5-5)

which is of the form:

$$\frac{d}{dt}\underline{x}^{[1]} = \left[A^{[1]}\right]\underline{x}^{[1]} + \left[B^{[2]}\right]\underline{x}^{[2]} + \underline{u}^{[1]} \qquad (5\text{-}6)$$

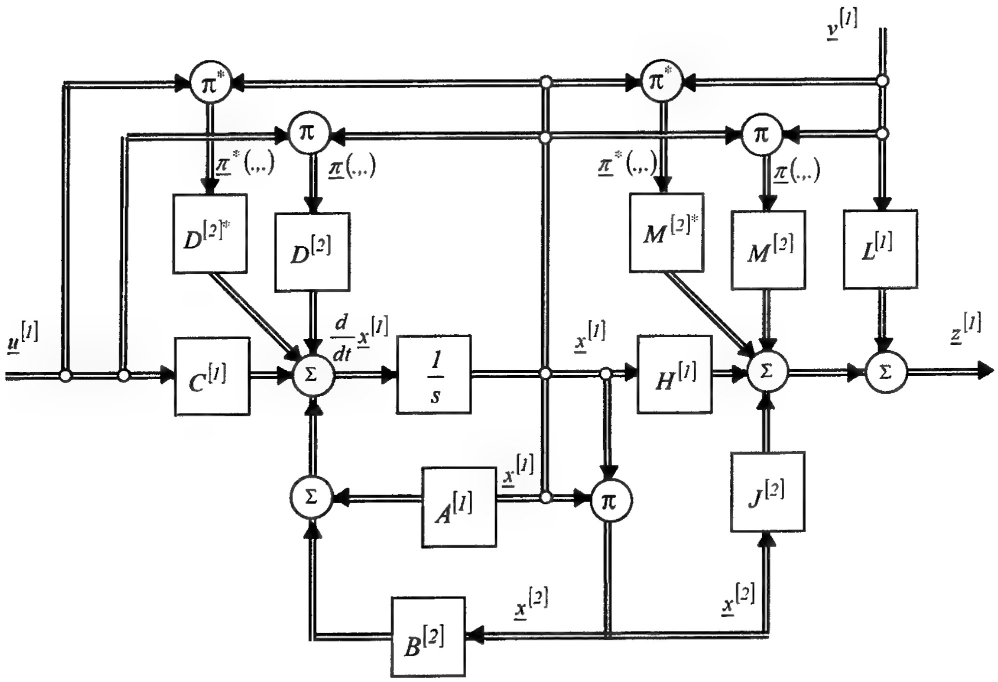

Figure 5.1 Block diagram of the forced-QBDS of equations (4-1), (5-1b)

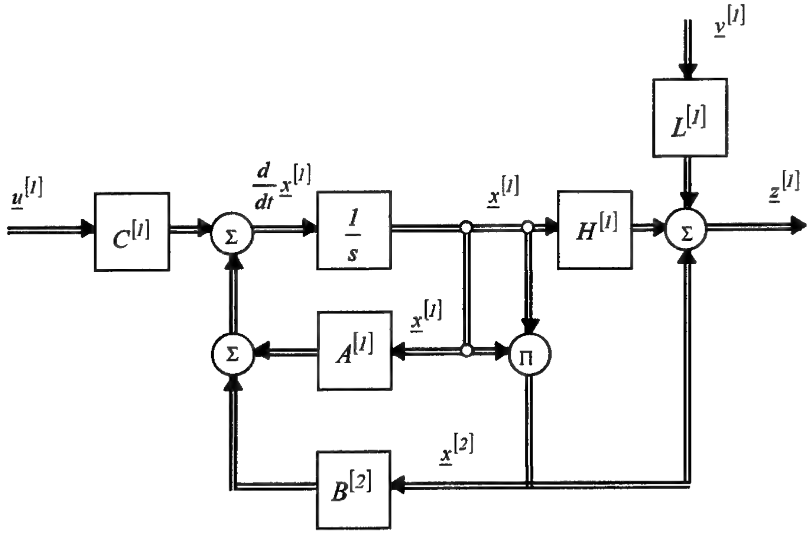

Figure 5.2 Block diagram of a forced-QBDS with linear input and disturbance

6. Extension to a Second Order Approximation of Analytical Functions

One obvious application of the quadratic/bilinear algebraic structure discussed in the previous sections is in the second order approximation of analytical functions. Non-linear dynamical systems with RHS that consists of analytical functions can be conveniently set up as a set of quadratic/bilinear differential equations [5]. In order to demonstrate this we consider the following non-linear dynamical system of the form:

$$\frac{d}{dt}x_i = f_i(\underline{x},\underline{u}); \quad i = 1,2,...,n \tag{6-1}$$

with the output model given by:

$$z_j = h_j(\underline{x},\underline{v}); \quad j = 1,2,...,r \tag{6-2}$$

Where:
$\underline{x} = \{x_i; i = 1,2,...,n\}$: is an $[n \times 1]$ linear state vector
$\underline{u} = \{u_i; i = 1,2,...,m \leq n\}$: is an $[m \times 1]$ linear input vector
$\underline{v} = \{v_i; i = 1,2,...,s\}$: is an $[s \times 1]$ output disturbance vector
$\underline{z} = \{z_i; i = 1,2,...,r\}$: is an $[r \times 1]$ output (measurement) vector

Assuming that functions $\{f_i(.,.)\}$ and $\{h_i(.,.)\}$ satisfy the usual conditions for the existence of the Taylor's series, it follows that:

$$\left.\frac{\partial f_i}{\partial \underline{x}}\right|_0 \delta\underline{x} = \sum_{j=1}^{n}\left.\left(\frac{\partial f_i}{\partial x_j}\right)\right|_0 \delta x_j$$

$$\left.\frac{\partial f_i}{\partial \underline{u}}\right|_0 \delta\underline{u} = \sum_{j=1}^{m}\left.\left(\frac{\partial f_i}{\partial u_j}\right)\right|_0 \delta u_j$$

$$\left.\frac{\partial}{\partial \underline{x}}\left(\frac{\partial f_i}{\partial \underline{x}}\delta\underline{x}\right)\right|_0 \delta\underline{x} = \sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left.\left(\sum_{j=1}^{n}\left(\frac{\partial f_i}{\partial x_j}\right)\delta x_j\right)\right|_0 \delta x_k = \sum_{k=1}^{n}\sum_{j=1}^{n}\left.\left(\frac{\partial^2 f_i}{\partial x_k \partial x_j}\right)\right|_0 \delta x_k \delta x_j$$

$$\left.\frac{\partial}{\partial \underline{x}}\left(\frac{\partial f_i}{\partial \underline{u}}\delta\underline{u}\right)\right|_0 \delta\underline{x} = \sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left.\left(\sum_{j=1}^{m}\left(\frac{\partial f_i}{du_j}\right)\delta u_j\right)\right|_0 \delta x_k = \sum_{k=1}^{n}\sum_{j=1}^{m}\left.\left(\frac{\partial^2 f_i}{\partial x_k \partial u_j}\right)\right|_0 \delta x_k \delta u_j$$

$$\left.\frac{\partial}{\partial \underline{u}}\left(\frac{\partial f_i}{\partial \underline{x}}\delta\underline{x}\right)\right|_0 \delta\underline{u} = \sum_{k=1}^{m}\frac{\partial}{\partial u_k}\left.\left(\sum_{j=1}^{n}\left(\frac{\partial f_i}{\partial x_j}\right)\delta x_j\right)\right|_0 \delta u_k = \sum_{k=1}^{m}\sum_{j=1}^{n}\left.\left(\frac{\partial^2 f_i}{\partial u_k dx_j}\right)\right|_0 \delta x_j \delta u_k$$

$$\left.\frac{\partial}{\partial \underline{u}}\left(\frac{\partial f_i}{\partial \underline{u}}\delta\underline{u}\right)\right|_0 \delta\underline{u} = \sum_{k=1}^{m}\frac{\partial}{\partial u_k}\left.\left(\sum_{j=1}^{m}\left(\frac{\partial f_i}{\partial u_j}\right)\delta u_j\right)\right|_0 \delta u_k = \sum_{k=1}^{m}\sum_{j=1}^{m}\left.\left(\frac{\partial^2 f_i}{\partial u_k \partial u_j}\right)\right|_0 \delta u_k \delta u_j$$

Since:

$$\sum_{k=1}^{n}\sum_{j=1}^{m}\left(\frac{\partial^2 f_i}{\partial x_k \partial u_j}\right)\Bigg|_0 \delta x_k \delta u_j = \sum_{k=1}^{m}\sum_{j=1}^{n}\left(\frac{\partial^2 f_i}{\partial u_k dx_j}\right)\Bigg|_0 \delta x_j \delta u_k$$

the second order Taylor's expansion gives:

$$\begin{aligned} f_i(\underline{x},\underline{u}) = f_i(\underline{x},\underline{u})\big|_o &+ \sum_{j=1}^{n}\left(\frac{\partial f_i}{\partial x_j}\right)\Bigg|_0 \delta x_j + \sum_{j=1}^{m}\left(\frac{\partial f_i}{\partial u_j}\right)\Bigg|_0 \delta u_j + \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial^2 f_i}{\partial x_k \partial x_j}\right)\Bigg|_0 \delta x_k \delta x_j \\ &+ \sum_{k=1}^{n}\sum_{j=1}^{m}\left(\frac{\partial^2 f_i}{\partial x_k \partial u_j}\right)\Bigg|_0 \delta x_k \delta u_j + \frac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m}\left(\frac{\partial^2 f_i}{\partial u_k \partial u_j}\right)\Bigg|_0 \delta u_k \delta u_j; \quad i = 1,2,\ldots,n \end{aligned} \tag{6-3}$$

Similarly:

$$\begin{aligned} h_i(\underline{x},\underline{v}) = h_i(\underline{x},\underline{v})\big|_o &+ \sum_{j=1}^{n}\left(\frac{\partial h_i}{\partial x_j}\right)\Bigg|_0 \delta x_j + \sum_{j=1}^{s}\left(\frac{\partial h_i}{\partial v_j}\right)\Bigg|_0 \delta v_j + \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial^2 h_i}{\partial x_k \partial x_j}\right)\Bigg|_0 \delta x_k \delta x_j \\ &+ \sum_{k=1}^{n}\sum_{j=1}^{s}\left(\frac{\partial^2 h_i}{\partial x_k \partial v_j}\right)\Bigg|_0 \delta x_k \delta v_j + \frac{1}{2}\sum_{k=1}^{s}\sum_{j=1}^{s}\left(\frac{\partial^2 h_i}{\partial v_k \partial v_j}\right)\Bigg|_0 \delta v_k \delta v_j; \quad i = 1,2,\ldots,r \end{aligned} \tag{6-4}$$

Furthermore since:

$$\sum_{k=1}^{n}\sum_{j=1}^{m}\left(\frac{\partial^2 f_i}{\partial x_k \partial u_j}\right)\Bigg|_0 \delta x_k \delta u_j = \sum_{k=1}^{n}\sum_{j=k}^{m}\alpha_{k,j}\left\{\left(\frac{\partial^2 f_i}{\partial x_k \partial u_j}\right)\Bigg|_0 \delta x_k \delta u_j + \left(\frac{\partial^2 f_i}{\partial x_j \partial u_k}\right)\Bigg|_0 \delta x_j \delta u_k\right\}$$

$$\alpha_{k,j} = \begin{cases} \frac{1}{2} & for \quad k = j \\ 1 & otherwise \end{cases}$$

It follows that equations (6-3) and (6-4) may be written as:

$$\begin{aligned} f_i(\underline{x},\underline{u}) = f_i(\underline{x},\underline{u})\big|_o &+ \sum_{j=1}^{n}\left(\frac{\partial f_i}{\partial x_j}\right)\Bigg|_0 \delta x_j + \sum_{j=1}^{m}\left(\frac{\partial f_i}{\partial u_j}\right)\Bigg|_0 \delta u_j + \sum_{k=1}^{n}\sum_{j=k}^{n}\alpha_{j,k}\left(\frac{\partial^2 f_i}{\partial x_k \partial x_j}\right)\Bigg|_0 \delta x_k \delta x_j \\ + \sum_{k=1}^{n}\sum_{j=k}^{m}\alpha_{k,j}&\left\{\left(\frac{\partial^2 f_i}{\partial x_k \partial u_j}\right)\Bigg|_0 \delta x_k \delta u_j + \left(\frac{\partial^2 f_i}{\partial x_j \partial u_k}\right)\Bigg|_0 \delta x_j \delta u_k\right\} + \sum_{k=1}^{m}\sum_{j=k}^{m}\alpha_{j,k}\left(\frac{\partial^2 f_i}{\partial u_k \partial u_j}\right)\Bigg|_0 \delta u_k \delta u_j \end{aligned} \tag{6-5}$$

$i = 1,2,\ldots,n$

And:

$$h_i(\underline{x},\underline{v}) = h_i(\underline{x},\underline{v})\Big|_o + \sum_{j=1}^{n}\left(\frac{\partial h_i}{\partial x_j}\right)\Bigg|_0 \delta x_j + \sum_{j=1}^{s}\left(\frac{\partial h_i}{\partial v_j}\right)\Bigg|_0 \delta v_j + \sum_{k=1}^{n}\sum_{j=k}^{n}\alpha_{k,j}\left(\frac{\partial^2 h_i}{\partial x_k \partial x_j}\right)\Bigg|_0 \delta x_k \delta x_j$$

$$+ \sum_{k=1}^{n}\sum_{j=k}^{m}\alpha_{k,j}\left\{\left(\frac{\partial^2 h_i}{\partial x_k \partial u_j}\right)\Bigg|_0 \delta x_k \delta v_j + \left(\frac{\partial^2 h_i}{\partial x_j \partial u_k}\right)\Bigg|_0 \delta x_j \delta v_k\right\} + \frac{1}{2}\sum_{k=1}^{s}\sum_{j=k}^{s}\alpha_{k,j}\left(\frac{\partial^2 h_i}{\partial v_k \partial v_j}\right)\Bigg|_0 \delta v_k \delta v_j \qquad (6\text{-}6)$$

$i = 1,2,\ldots,r$

6.1. System Dynamical Model

Making the following substitutions:

$$a_{i,j} = \left(\frac{\partial f_i}{\partial x_j}\right); \quad j = 1,2,\ldots,n; \quad i = 1,2,\ldots,n\,; \; c_{i,j} = \left(\frac{\partial f_i}{\partial u_j}\right); \quad j = 1,2,\ldots,m; \quad i = 1,2,\ldots,n$$

$$b_{i,l} = \left(\frac{\partial^2 f_i}{\partial x_k \partial x_j}\right); \quad l = 1,2,\ldots,\frac{n(n+1)}{2}; \quad i = 1,2,\ldots,n\,; \; l = (k-1)\left(m - \frac{k}{2}\right) + j$$

$$s_{k,j}^{[i]} = \left(\frac{\partial^2 f_i}{\partial x_k \partial u_j}\right); \quad j = 1,2,\ldots,m; \quad k = j, j+1,\ldots,n$$

$$s_{j,k}^{[i]} = \left(\frac{\partial^2 f_i}{\partial x_j \partial u_k}\right); \quad j = 1,2,\ldots,m; \quad k = j, j+1,\ldots,n$$

$$g_{i,l} = \left(\frac{\partial^2 f_i}{\partial u_k \partial u_j}\right); l = 1,2,\ldots,\frac{m(m+1)}{2}; i = 1,2,\ldots,n\,; \; l = (k-1)\left(m - \frac{k}{2}\right) + j$$

Using the relationships of equations (6-3), (6-1), The second order perturbation model (differential equation) for system dynamics may be written as:

$$\frac{d}{dt}\delta x_i = \sum_{j=1}^{n} a_{i,j}\delta x_j + \sum_{j=1}^{m} c_{i,j}\delta u_j + \sum_{k=1}^{n}\sum_{j=k}^{n}\alpha_{j,k} b_{i,\mu}\delta x_k \delta x_j + \sum_{k=1}^{n}\sum_{j=k}^{m}\alpha_{k,j}\left\{s_{k,j}^{[i]}\delta x_k \delta u_j + s_{j,k}^{[i]}\delta x_j \delta u_k\right\}$$

$$+ \sum_{k=1}^{m}\sum_{j=k}^{m}\alpha_{j,k} g_{i,\eta}\delta u_k \delta u_j; \quad i = 1,2,\ldots,n \qquad (6\text{-}7)$$

Where:

$$\mu = (k-1)\left(n - \frac{k}{2}\right) + j; \quad \eta = (k-1)\left(m - \frac{k}{2}\right) + j$$

The first two terms on the RHS of equation (6-7) are linear in $\{\underline{\delta x}\}$ and $\{\underline{\delta u}\}$ respectively, while the third and fifth terms are quadratic involving $\left\{\underline{\delta x}^{[2]}\right\}$ and $\left\{\underline{\delta u}^{[2]}\right\}$ respectively. The fourth term is the bilinear state-input term in $\{\underline{\delta x}, \underline{\delta v}\}$. Utilising the results of previous sections, equation (6-7) (after some algebraic manipulations) may be written, in state-space form, as:

$$\frac{d}{dt}\underline{\delta x}^{[1]} = \left[A^{[1]}\right]\underline{\delta x}^{[1]} + \left[B^{[2]}\right]\underline{\delta x}^{[2]} + \left[C^{[1]}\right]\underline{\delta u}^{[1]} + \underline{\Pi}^{[2]}\left(\underline{\delta x}^{[1]},\underline{\delta u}^{[1]}\right) + \left[G^{[2]}\right]\underline{\delta u}^{[2]} \quad (6\text{-}8)$$

Or alternatively as:

$$\begin{aligned}\frac{d}{dt}\underline{\delta x}^{[1]} &= \left[A^{[1]}\right]\underline{\delta x}^{[1]} + \left[B^{[2]}\right]\underline{\delta x}^{[2]} + \left[C^{[1]}\right]\underline{\delta u}^{[1]} + \left[D^{[2]}\right]\underline{\pi}^{[2]}\left(\underline{\delta x}^{[1]},\underline{\delta u}^{[1]}\right) \\ &\quad + \left[D^{[2]^*}\right]\underline{\pi}^{[2]^*}\left(\underline{\delta x}^{[1]},\underline{\delta u}^{[1]}\right) + \left[G^{[2]}\right]\underline{\delta u}^{[2]}\end{aligned} \quad (6\text{-}9)$$

6.2. System Output (Measurement) Model

Making the following substitutions:

$$h_{i,j} = \left(\frac{\partial h_i}{\partial x_j}\right); \quad j = 1,2,\ldots,n; \quad i = 1,2,\ldots,r\,; \; l_{i,j} = \left(\frac{\partial h_i}{\partial v_j}\right); \quad j = 1,2,\ldots,s; \quad i = 1,2,\ldots,r$$

$$j_{i,l} = \left(\frac{\partial^2 h_i}{\partial x_k \partial x_j}\right); \quad l = 1,2,\ldots,\frac{n(n+1)}{2}; \quad i = 1,2,\ldots,r\,; \; l = (k-1)\left(m-\frac{k}{2}\right)+j$$

$$t_{k,j}^{[i]} = \left(\frac{\partial^2 h_i}{\partial x_k \partial v_j}\right); \quad j = 1,2,\ldots,s; \quad k = j, j+1,\ldots,n$$

$$t_{j,k}^{[i]} = \left(\frac{\partial^2 h_i}{\partial x_j \partial v_k}\right); \quad j = 1,2,\ldots,s; \quad k = j, j+1,\ldots,n$$

$$n_{i,l} = \left(\frac{\partial^2 f_i}{\partial v_k \partial v_j}\right); l = 1,2,\ldots,\frac{s(s+1)}{2}; i = 1,2,\ldots,r\,; \; l = (k-1)\left(s-\frac{k}{2}\right)+j$$

Using the relationships established in equations (6-3) and (6-1), the second order perturbation model (measurement model) for the output may be written as:

$$\begin{aligned}\delta z_i &= \sum_{j=1}^{n} h_{i,j}\delta x_j + \sum_{j=1}^{s} l_{i,j}\delta v_j + \sum_{k=1}^{n}\sum_{j=k}^{n} \alpha_{j,k} j_{i,\mu}\delta x_k \delta x_j + \sum_{k=1}^{n}\sum_{j=k}^{s} \alpha_{k,j}\left\{t_{k,j}^{[i]}\delta x_k \delta v_j + t_{j,k}^{[i]}\delta x_j \delta v_k\right\} \\ &\quad + \sum_{k=1}^{s}\sum_{j=k}^{s} \alpha_{j,k} n_{i,\eta}\delta v_k \delta v_j; \quad i = 1,2,\ldots,r\end{aligned} \quad (6\text{-}10)$$

Where:

$$\mu = (k-1)\left(n-\frac{k}{2}\right)+j; \quad \eta = (k-1)\left(s-\frac{k}{2}\right)+j$$

The first two terms on the RHS of equation (6-10) are linear in $\{\underline{\delta x}\}$ and $\{\underline{\delta v}\}$ respectively, while the third and fifth terms are quadratic involving $\left\{\underline{\delta x}^{[2]}\right\}$ and $\left\{\underline{\delta v}^{[2]}\right\}$

respectively. The fourth term is the bilinear state-input term in $\{\underline{\delta x}, \underline{\delta v}\}$. Utilising the results of previous sections, equation (6-10) (after some algebraic manipulations) may be written, in state-space form, as:

$$\underline{\delta z}^{[1]} = \left[H^{[1]}\right]\underline{\delta x}^{[1]} + \left[J^{[2]}\right]\underline{\delta x}^{[2]} + \left[L^{[1]}\right]\underline{\delta v}^{[1]} + \underline{\Pi}^{[2]}\left(\underline{\delta x}^{[1]}, \underline{\delta v}^{[1]}\right) + \left[G^{[2]}\right]\underline{\delta v}^{[2]} \tag{6-11}$$

Or alternatively as:

$$\begin{aligned}\underline{\delta z}^{[1]} = {} & \left[H^{[1]}\right]\underline{\delta x}^{[1]} + \left[J^{[2]}\right]\underline{\delta x}^{[2]} + \left[L^{[1]}\right]\underline{\delta v}^{[1]} + \left[M^{[2]}\right]\underline{\pi}^{[2]}\left(\underline{\delta x}^{[1]}, \underline{\delta v}^{[1]}\right) \\ & + \left[M^{[2]^*}\right]\underline{\pi}^{[2]^*}\left(\underline{\delta x}^{[1]}, \underline{\delta v}^{[1]}\right) + \left[N^{[2]}\right]\underline{\delta u}^{[2]}\end{aligned} \tag{6-12}$$

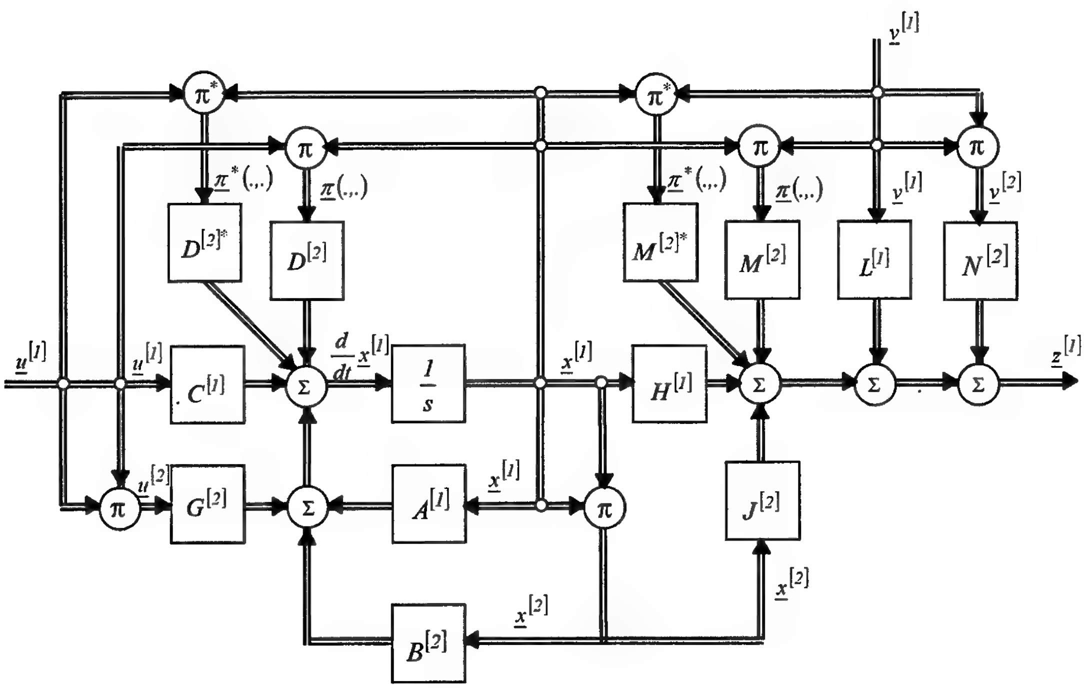

Figure 6.1 Block Diagram of the Forced-QBDS of equations (6-9), (6-12)

7. Conclusion

In this report, formal definitions of free and forced quadratic/bilinear dynamical systems were given and the algebraic structure of this class of systems was explored with a view to setting up a systematic approach for deriving state dynamics and measurement models. This class of dynamical systems is characterized by a set of first order differential equations with the RHS that contains linear and quadratic terms in system states as well as bilinear terms involving state and input (or control) variables. An example of the quadratic state-space model of the airframe aerodynamics is given in Appendix-A. Properties of the quadratic and bilinear vectors were investigated and relationships between these and linear vectors established. Systematic procedure for constructing quadratic state and the bilinear state-input vectors was derived. The quadratic algebraic structure was applied to the formulation of a state-space model for a second order approximation of a general (analytic) non-linear system.

The concept of quadratic and bilinear generator matrices was introduced that allows linear (state and state-input) vectors to be mapped onto the quadratic and bilinear vectors. Properties of these generator matrices were explored and it was shown that inverse generator matrices may be defined (in all but one case) that allows quadratic and bilinear vectors to be mapped onto linear state and linear input vectors. State-space representation of the system output or measurement model was also derived for the case where this contains linear and quadratic terms in states and bilinear terms in state-disturbance (noise) variables.

The state-space structure of quadratic/bilinear dynamical systems considered in this report should facilitate analysis of this class of non-linear systems and possibly lead to general synthesis techniques or extension of linearised techniques for applications to these problems.

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Appendix-A

Euler's Equations of Motion

The six equations of motion for a body with six degrees of freedom may be written as [Ref. 5]:

$$
\begin{aligned}
m(\dot{u} + wq - vr) &= X + T + g_x m \\
m(\dot{v} + ur - wp) &= Y + g_y m \qquad \text{(A1-1)} \\
m(\dot{w} - uq + vp) &= Z + g_x m
\end{aligned}
$$

$$
\begin{aligned}
I_{xx}\dot{p} - (I_{yy} - I_{zz})qr + I_{yz}(r^2 - q^2) - I_{zx}(pq + \dot{r}) + I_{xy}(rp - \dot{q}) &= L \\
I_{yy}\dot{q} - (I_{zz} - I_{xx})rp + I_{zx}(p^2 - r^2) - I_{xy}(qr + \dot{p}) + I_{yz}(pq - \dot{r}) &= M \qquad \text{(A1-2)} \\
I_{zz}\dot{r} - (I_{xx} - I_{yy})pq + I_{xy}(q^2 - p^2) - I_{yz}(rp + \dot{q}) + I_{zx}(qr - \dot{p}) &= N
\end{aligned}
$$

Separating the derivative terms and after some algebraic manipulation, equations (A1-1), (A1-2) may be written in a vector form as:

$$
\frac{d}{dt}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} uq \\ ur \\ vp \\ vr \\ wp \\ wq \end{bmatrix} + \begin{bmatrix} \tilde{X} + \tilde{T} \\ \tilde{Y} \\ \tilde{Z} \end{bmatrix} + \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} \qquad \text{(A1-3)}
$$

$$
\frac{d}{dt}\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \left[[A]^{-1}[B]\right] \begin{bmatrix} p^2 \\ pq \\ pr \\ q^2 \\ qr \\ r^2 \end{bmatrix} + \left[[A]^{-1}\right] \begin{bmatrix} L \\ M \\ N \end{bmatrix} \qquad \text{(A1-4)}
$$

Note that the states (u,v,w,p,q,r) appear as quadratic terms in equations (A 1-3), (A1-4).

Where:

$$\tilde{X} = \frac{X}{m}; \quad \tilde{Y} = \frac{Y}{m}; \quad \tilde{Z} = \frac{Z}{m}; \quad \tilde{T} = \frac{T}{m}.$$

$$[A]=\begin{bmatrix} I_{xx} & -I_{xy} & -I_{zx} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{yz} & I_{zz} \end{bmatrix}$$

$$[B]=\begin{bmatrix} 0 & I_{zx} & -I_{xy} & I_{yz} & \left(I_{yy}-I_{zz}\right) & -I_{yz} \\ -I_{zx} & -I_{yz} & \left(I_{zz}-I_{xx}\right) & 0 & I_{xy} & I_{zx} \\ I_{xy} & \left(I_{xx}-I_{yy}\right) & I_{yz} & -I_{xy} & -I_{zx} & 0 \end{bmatrix}$$

$$[A]^{-1}=\frac{1}{\Delta}\begin{bmatrix} \left(I_{yy}I_{zz}-I_{yz}^{2}\right) & \left(I_{zz}I_{xy}+I_{yz}I_{zx}\right) & \left(I_{yz}I_{xy}+I_{yy}I_{zx}\right) \\ \left(I_{zz}I_{xy}+I_{yz}I_{zx}\right) & \left(I_{xx}I_{zz}-I_{zx}^{2}\right) & \left(I_{xx}I_{yz}+I_{zx}I_{xy}\right) \\ \left(I_{yz}I_{xy}+I_{yy}I_{zx}\right) & \left(I_{xx}I_{yz}+I_{xy}I_{zx}\right) & \left(I_{xx}I_{yy}-I_{xy}^{2}\right) \end{bmatrix}$$

$$\Delta=\left(I_{xx}I_{yy}I_{zz}-I_{xx}I_{yz}^{2}-I_{yy}I_{zx}^{2}-I_{zz}I_{xy}^{2}-2I_{yz}I_{zx}I_{xy}\right)$$

Combining equations (A1-3) and (A1-4), we obtain the full 6th order rigid body dynamics state equations as:

$$\frac{d}{dt}\begin{bmatrix} \underline{x}_{1}^{[1]} \\ \hdashline \underline{x}_{2}^{[1]} \end{bmatrix}=\left[\begin{array}{c:c} [C] & [0] \\ \hdashline [0] & \left[[A]^{-1}[B]\right] \end{array}\right]\begin{bmatrix} \underline{x}_{1}^{[2]} \\ \hdashline \underline{x}_{2}^{[2]} \end{bmatrix}+\left[\begin{array}{c:c} [I] & [0] \\ \hdashline [0] & \left[[A]^{-1}\right] \end{array}\right]\begin{bmatrix} \underline{u}_{1}^{[1]} \\ \hdashline \underline{u}_{2}^{[1]} \end{bmatrix}+\begin{bmatrix} \underline{g} \\ \hdashline \underline{0} \end{bmatrix} \qquad \text{(A1-5)}$$

Where:

$$[C]=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{x}_{1}^{[1]}=[u \quad v \quad w]^{T};\ \underline{x}_{2}^{[1]}=[p \quad q \quad r]^{T}$$

$$\underline{x}_{1}^{[2]}=[uq \quad ur \quad vp \quad vr \quad wp \quad wq]^{T};\ \underline{x}_{2}^{[2]}=\left[p^{2} \quad pq \quad pr \quad q^{2} \quad qr \quad r^{2}\right]^{T}$$

$$\underline{u}_{1}^{[1]}=\left[\tilde{X}+\tilde{T} \quad \tilde{Y} \quad \tilde{Z}\right]^{T};\ \underline{u}_{2}^{[1]}=[L \quad M \quad N]^{T};\ \underline{g}=\left[g_{x} \quad g_{y} \quad g_{z}\right]^{T}$$

Equation (A1-5) may be written in a compact form as:

$$\frac{d}{dt}\underline{x}^{[1]}=[F]\underline{x}^{[2]}+[G]\underline{u}^{[1]}+g^{[1]}$$

Where:

$$[F]=\left[\begin{array}{c:c} [C] & [0] \\ \hdashline [0] & \left[[A]^{-1}[B]\right] \end{array}\right]$$: is the 6x12 quadratic state coefficient matrix

$$[G] = \left[\begin{array}{c:c} [I] & [0] \\ \hdashline [0] & \left[[A]^{-1}\right] \end{array}\right]$$: is the 6x6 input coefficient matrix

$\underline{x}^{[1]} = \left[\underline{x}_1^{[1]} \;\vdots\; \underline{x}_2^{[1]}\right]^T = \left[u \quad v \quad w \;\vdots\; p \quad q \quad r\right]^T$: is the 6x1 linear-state vector

$\underline{x}^{[2]} = \left[\underline{x}_1^{[2]} \;\vdots\; \underline{x}_2^{[2]}\right]^T = \left[uq \quad ur \quad vp \quad vr \quad wp \quad wq \;\vdots\; p^2 \quad pq \quad pr \quad q^2 \quad qr \quad r^2\right]^T$: is the 12x1 quadratic-state vector

$\underline{u}^{[1]} = \left[\underline{u}_1^{[1]} \;\vdots\; \underline{u}_1^{[1]}\right]^T = \left[\tilde{X} + \tilde{T} \quad \tilde{Y} \quad \tilde{Z} \;\vdots\; L \quad M \quad N\right]^T$: is 6x1 a vector function of control inputs, forces and moments

$\underline{g}^{[1]} = \left[\underline{g} \;\vdots\; \underline{0}\right]^T = \left[g_x \quad g_y \quad g_z \quad 0 \;\vdots\; 0 \quad 0\right]^T$: is the 6x1 gravity (or disturbance) vector

Appendix-B

$$
\left[x^{[2]}_{[1]}\right]=\begin{bmatrix}
x_1 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
\frac{x_2}{\sqrt{2}} & \frac{x_1}{\sqrt{2}} & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
\frac{x_3}{\sqrt{2}} & 0 & \frac{x_1}{\sqrt{2}} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{x_{n-1}}{\sqrt{2}} & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \frac{x_1}{\sqrt{2}} & 0 \\
\frac{x_n}{\sqrt{2}} & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \frac{x_1}{\sqrt{2}} \\
0 & x_2 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
0 & \frac{x_3}{\sqrt{2}} & \frac{x_2}{\sqrt{2}} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
0 & \frac{x_4}{\sqrt{2}} & 0 & \frac{x_2}{\sqrt{2}} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \frac{x_{n-1}}{\sqrt{2}} & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \frac{x_2}{\sqrt{2}} & 0 \\
0 & \frac{x_n}{\sqrt{2}} & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \frac{x_2}{\sqrt{2}} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & 0 & x_r & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & \cdots & \cdots & 0 & \frac{x_{r+1}}{\sqrt{2}} & \frac{x_r}{\sqrt{2}} & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & \cdots & \cdots & 0 & \frac{x_{r+2}}{\sqrt{2}} & 0 & \frac{x_r}{\sqrt{2}} & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & 0 & \frac{x_{n-1}}{\sqrt{2}} & 0 & 0 & 0 & \cdots & \frac{x_r}{\sqrt{2}} & 0 \\
0 & \cdots & \cdots & \cdots & 0 & \frac{x_n}{\sqrt{2}} & 0 & 0 & 0 & \cdots & 0 & \frac{x_r}{\sqrt{2}} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_{n-1} & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \frac{x_n}{\sqrt{2}} & \frac{x_{n-1}}{\sqrt{2}} \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & x_n
\end{bmatrix} \qquad \text{(B1-1)}
$$

: is the $\left[\frac{n(n+1)}{2} x\, n\right]$ n-state quadratic generator matrix (*qgm*).

$$\left[\overline{X}^{[2]}_{[1]}\right]_{nxl} = \begin{bmatrix} x_1 & 0 & 0 & \dots & \dots & 0 \\ \frac{x_2}{\sqrt{2}} & \frac{x_1}{\sqrt{2}} & 0 & \dots & \dots & 0 \\ \frac{x_3}{\sqrt{2}} & 0 & \frac{x_1}{\sqrt{2}} & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{x_m}{\sqrt{2}} & 0 & \dots & \dots & \dots & \frac{x_1}{\sqrt{2}} \\ \frac{x_{m+1}}{\sqrt{2}} & 0 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{x_n}{\sqrt{2}} & 0 & 0 & \dots & \dots & 0 \\ 0 & x_2 & 0 & \dots & \dots & 0 \\ 0 & \frac{x_3}{\sqrt{2}} & \frac{x_2}{\sqrt{2}} & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots \\ 0 & \frac{x_m}{\sqrt{2}} & 0 & 0 & \dots & \frac{x_2}{\sqrt{2}} \\ 0 & \frac{x_{m+1}}{\sqrt{2}} & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \frac{x_n}{\sqrt{2}} & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & x_m \\ 0 & \dots & \dots & \dots & 0 & \frac{x_{m+1}}{\sqrt{2}} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 & \frac{x_n}{\sqrt{2}} \end{bmatrix} \quad \text{(B1-2)}$$

: is the $\left[\frac{m(2n-m+1)}{2} \, x \, m\right]$ n-state sum bilinear generator matrix (*sbgm*).

$$\left[\overline{U}_{[1]}^{[2]}\right]_{nxp} = \begin{bmatrix}
u_1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\frac{u_2}{\sqrt{2}} & \frac{u_1}{\sqrt{2}} & 0 & \cdots & \cdots & \cdots & \cdots & \cdot\cdot & \cdots & 0 \\
\frac{u_3}{\sqrt{2}} & 0 & \frac{u_1}{\sqrt{2}} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{u_m}{\sqrt{2}} & 0 & \cdots & \cdots & 0 & \frac{u_1}{\sqrt{2}} & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & \frac{u_1}{\sqrt{2}} & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \frac{u_1}{\sqrt{2}} \\
0 & u_2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \frac{u_3}{\sqrt{2}} & \frac{u_2}{\sqrt{2}} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \frac{u_m}{\sqrt{2}} & 0 & \cdots & 0 & \frac{u_2}{\sqrt{2}} & 0 & \cdots & \cdots & \cdots \\
\cdots & 0 & \cdots & \cdots & \cdots & 0 & \cdots & 0 & \frac{u_2}{\sqrt{2}} & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \frac{u_2}{\sqrt{2}} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & 0 & u_m & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & \frac{u_m}{\sqrt{2}} & 0 & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \frac{u_m}{\sqrt{2}} & 0 \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \frac{u_m}{\sqrt{2}}
\end{bmatrix} \tag{B1-3}$$

: is the $\left[\frac{m(2n-m+1)}{2} \, x \, n\right]$ m-input sum bilinear generator matrix (*sbgm*).

$$\left[\overline{X}^{[2]^*}_{[1]}\right]_{nx1} = \begin{bmatrix} -\frac{x_2}{\sqrt{2}} & \frac{x_1}{\sqrt{2}} & 0 & \dots & \dots & 0 \\ -\frac{x_3}{\sqrt{2}} & 0 & \frac{x_1}{\sqrt{2}} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\frac{x_m}{\sqrt{2}} & 0 & \dots & \dots & 0 & \frac{x_1}{\sqrt{2}} \\ -\frac{x_{m+1}}{\sqrt{2}} & 0 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\frac{x_n}{\sqrt{2}} & 0 & \dots & \dots & \dots & 0 \\ 0 & -\frac{x_3}{\sqrt{2}} & \frac{x_2}{\sqrt{2}} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & -\frac{x_m}{\sqrt{2}} & 0 & \dots & 0 & \frac{x_2}{\sqrt{2}} \\ 0 & -\frac{x_{m+1}}{\sqrt{2}} & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & -\frac{x_n}{\sqrt{2}} & 0 & \dots & \dots & 0 \\ \dots & 0 & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 & -\frac{x_{m+1}}{\sqrt{2}} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & -\frac{x_n}{\sqrt{2}}. \end{bmatrix} \tag{B1-4}$$

: is the $\left[\frac{m(2n-m-1)}{2} \; x \; m\right]$ n-state difference bilinear generator matrix (*dbgm*).

$$\left[\overline{U}^{[2]^{*}}_{[1]}\right]_{nxp} = \begin{bmatrix}
\frac{u_2}{\sqrt{2}} & -\frac{u_1}{\sqrt{2}} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\frac{u_3}{\sqrt{2}} & 0 & -\frac{u_1}{\sqrt{2}} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{u_m}{\sqrt{2}} & 0 & \cdots & \cdots & 0 & -\frac{u_1}{\sqrt{2}} & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & -\frac{u_1}{\sqrt{2}} & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -\frac{u_1}{\sqrt{2}} \\
0 & \frac{u_3}{\sqrt{2}} & -\frac{u_2}{\sqrt{2}} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \frac{u_m}{\sqrt{2}} & 0 & \cdots & 0 & -\frac{u_2}{\sqrt{2}} & 0 & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -\frac{u_2}{\sqrt{2}} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & -\frac{u_m}{\sqrt{2}} & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -\frac{u_m}{\sqrt{2}} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -\frac{u_m}{\sqrt{2}}
\end{bmatrix}. \qquad \text{(B1-5)}$$

: is the $\left[\frac{m(2n-m-1)}{2}\, x\, n\right]$ m-input difference bilinear generator matrix (*dbgm*).

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19. ABSTRACT

A number of important control and estimation problems in the field of aerospace and avionics involve dynamical models that are quadratic and bilinear functions of system states and inputs. In this report, formal definitions of free and forced quadratic/bilinear dynamical systems are given and the algebraic structure of this class of systems is explored with a view to setting up a systematic approach for deriving state and measurement models. Properties of quadratic and bilinear vectors are investigated and relationships between these and linear vectors established. Systematic procedure for constructing quadratic state and bilinear state-input vectors is derived. The quadratic/bilinear vector modeling technique is applied to the formulation of a state-space model for a second order approximation of a general non-linear system.